Cal Poly Winter 2015	**CSC 141** Discrete Structures	Eriq Augustine

Exam 02 (Take-Home)
March 03^{th} - March 05^{th}, 2015

Name: KEY **Username:** KEY

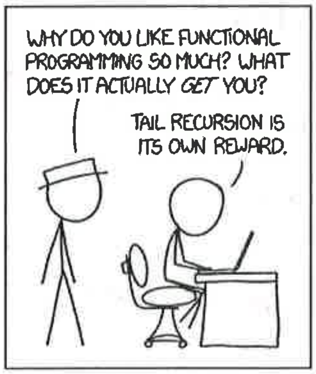
Question:	1	2	3	4	5	XC	Total
Points:	15	10	10	20	40	5*	95
Score:							

There are 6 problems on 11 sheets (counting this cover sheet). Each problem is divided into a number of questions.

Do all of your work in this exam, and cross out any work that should be ignored. You may use the back of a page, but clearly indicate the problem that your work refers to. This exam is a take-home, so you may reference your notes and the book. However, all work and thinking is to be done **alone**. Please refrain from communicating with other students or using the internet.

State any assumptions that you make. If you think that there is an error/typo, then ask via email or during office hours. The questions are not necessarily ordered by difficulty.

Do not start until class is finished. This exam is due at the beginning of class on Thursday, March 05^{th}.



Mean : 83.26
Median : 86.32
Max : 105.26

Relative to M1
Mean : + 02.08
Median : + 00.93
Max : - 00.19

1 Sequences

1. Find a recursive definition for the following sequences:

   (a) 2, 6, 18, 54, ...
   4 12 36

   $a_n = a_{n-1} \cdot 3, \ a_1 = 2$

   (b) 3, 13, 53, 213, ...
   3 12 52 212

   $a_n = a_{n-1} \cdot 4 + 1, \ a_1 = 3$

   (c) $a_n = \frac{n+1}{3}, n > 0$

   n: 1 2 3 4
   $2/3$ $3/3$ $4/3$ $5/3$

   $a_n = a_{n-1} + 1/3, \ a_1 = 2/3$

2. Find the closed equation for: $a_n = 6a_{n-1} - 1$ with $a_1 = 15$? Clearly show all your work. You may leave summations in your final answer if you so choose.

$a_n = 6a_{n-1} - 1, \ a_1 = 15$

$a_1 = 15$

$a_2 = 6(a_{n-1}) - 1 \quad = 6^1 a_1 - 1$

$a_3 = 6(6a_{n-1} - 1) - 1 \quad = 6^2 a_1 - 6 - 1 = 6^2 a_1 - (6+1)$

$a_4 = 6(6(6a_1 - 1) - 1) - 1 \quad = 6^3 a_1 - 36 - 6 - 1 = 6^3 a_1 - (36+6+1)$

$= 6^3 a_1 - (6^2 + 6^1 + 6^0)$

$= 6^{n-1} a_1 - \sum_{i=0}^{n-2} 6^i$

$a_n =$

$= \boxed{6^{n-1}(15) - \sum_{i=0}^{n-2} 6^i}$

2 Matrices

Given the following definitions of A and B, compute the given expressions.

$$A = \begin{bmatrix} 5 & 2 \\ 3 & -6 \\ -2 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 5 \\ 8 & -4 \end{bmatrix}$$

1. AB

$$\begin{bmatrix} 5 & 2 \\ 3 & -6 \\ -2 & 9 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 8 & -4 \end{bmatrix} = \begin{bmatrix} 21 & 17 \\ -45 & 39 \\ 70 & -46 \end{bmatrix}$$

2. BA

$$\begin{bmatrix} 1 & 5 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 3 & -6 \\ -2 & 9 \end{bmatrix} \quad \times$$

Can't do, # of cols of LHS must equal # of rows of RHS.

3. B^t

$$\begin{bmatrix} 1 & 5 \\ 8 & -4 \end{bmatrix}^t = \begin{bmatrix} 1 & 8 \\ 5 & -4 \end{bmatrix}$$

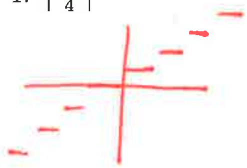
4. BB^t

$$\begin{bmatrix} 1 & 5 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} 1 & 8 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 26 & -12 \\ -12 & 80 \end{bmatrix}$$

3 Functions

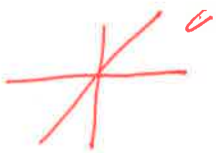
Determine if each of the following from $Z \to Z$ (all integers) is an **injection**, **surjection**, **bijection**, or **none**. Choose the classification that most accurately describes the function.

1. $\lceil \frac{3x}{4} \rceil$



Surjection

2. $2x + 3$

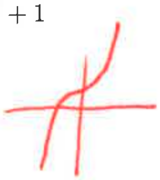


injection. Since $Z \to Z$, some ints get skipped.

$x = 1,\ 2+3=5$ 6 skipped

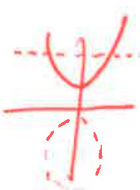
$x = 2\ \ 4+3=7$

3. $x^3 + 1$



injection Same as 2.

4. $x^2 + 1$



none

4 Growth of Functions

For all functions in this problem, assume the domain is Z^+.

1. Find the big-θ for the following functions. (You may solve these informally if you so choose).

   (a) $\frac{2^n}{4}$

   $\Theta(2^n)$

   (b) 8000^{33}

   $\Theta(1)$

   1 and 8000^{33} grow at the same speed.

   (c) $5n + 5\log n$

   $\Theta(n)$

   n grows faster than log n

   (d) $6n^4 - 2n!$

   $\Theta(n!)$

   Notes:
   - n! grows faster than n^4
   - Domain = Z^+
   - Θ not O

   so we need to model function, not bound it.

2. Formally find the big-O ($g(n)$ and witnesses) for the following functions. Show all your work and keep your $g(n)$ and witnesses reasonable. (So you may not just assume a $g(n)$ of $n!^{n!}$ and call it done.)

(a) $(n^3 - 2)(n^2 + 4)$

$\cancel{(n^3+2}$

$(n^3-2)(n^2+4)$

$n^5+4n^3-2n^2-8$

$n^5+4n^5+2n^5+n^5$

$8n^5$

$$\boxed{\begin{array}{l} g(x) = n^5 \\ C = 8 \\ k = 1 \end{array}}$$

$n^5 \geq n^5$

$4n^5 \geq 4n^3$ as long as $n \geq 0$

$2n^5 \geq -2n^2$ " " " "

$n^5 \geq -8$ " " " "

($n^5 \geq 8$ when $n \geq 2$)

so let $k = 1$

(b) $n^4 + (1 - n)^3 + 16$

$n^4 + (1-n)^3 + 16$

$n^4 - n^3 + 3n^2 - 3n + 1 + 16$

$n^4 - n^3 + 3n^2 - 3n + 17$

$n^4 + n^4 + 3n^4 + 3n^4 + n^4$

$9n^4$

$n^4 \geq n^4$

$n^4 \geq -n^3$

$3n^4 \geq 3n^2$

$3n^4 \geq -3n$

$n^4 \geq 17$ when $n \geq 3$

so let $k = 3$
(could let it be 2 as well.)

$$\boxed{\begin{array}{l} g(n) = n^4 \\ C = 9 \\ k = 3 \end{array}}$$

$P(n)$

5 Proofs

1. Prove that $\underbrace{1+4+7+10+...+(3n-2) = \frac{n(3n-1)}{2}}_{P(n)}$ where $n > 0$. Show all your work clearly state your conclusion.

Use Induction.

Base Step: $P(1)$

$P(1) = 3(1)-2 = \frac{(1)(3(1)-1)}{2}$

$3-2 = \frac{3-1}{2}$

$1 = 1$ ✓

Inductive Step:

Assume $P(k) = 1+4+7+...+(3k-2) = \frac{k(3k-1)}{2}$

Prove $P(k) \rightarrow P(k-1)$

Direct ~~Proo~~. Proof

$1+4+...+(3k-2)+(3(k+1)-2) = \frac{k(3k-1)}{2} + (3(k+1)-2)$

$= \frac{3k^2-k}{2} + 3k+3-2$

$= \frac{3k^2-k}{2} + \frac{2(3k+1)}{2}$

$= \frac{3k^2-k+6k+2}{2}$

$= \frac{(k+1)(3k+2)}{2}$

$= \frac{(k+1)(3(k+1)-1)}{2}$ ✓

Conclusion: when adding the k+1 term to both sides in P(k), we found the result of P(k+1) ✓

$P(n)$

2. Prove that if $A_1, A_2, ..., A_n$ and B are sets, then

$$(A_1 \cap A_2 \cap ... \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap ... \cap (A_n \cup B)$$

Show all your work clearly state your conclusion.

Note: we are proving the general case of the Distributive Law for sets.

Induction

Base Step:

$P(1)$

$A_1 \cup B = A_1 \cup B$

OR

$P(2)$

$(A_1 \cap A_2) \cup B = (A_1 \cup B) \cap (A_2 \cup B)$

↑ Definition of Distributive Law

Inductive Step

Assume $P(k)$: $(A_1 \cap A_2 \cap ... \cap A_k) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap ... \cap (A_k \cup B)$

Prove $P(k) \to P(k+1)$

Direct

LHS of $P(k+1)$: $(A_1 \cap A_2 \cap ... \cap A_k \cap A_{k+1}) \cup B$

Treat as one set

Associative Law: $[(A_1 \cap A_2 \cap ... \cap A_k) \cap A_{k+1}] \cup B$

Def of Distributive Law: $[(A_1 \cap A_2 \cap ... \cap A_k) \cup B] \cap (A_{k+1} \cup B)$

Def of $P(k)$: $[(A_1 \cup B) \cap (A_2 \cup B) \cap ... \cap (A_k \cup B)] \cap (A_{k+1} \cup B)$

Associative Law: $(A_1 \cup B) \cap (A_2 \cup B) \cap ... \cap (A_k \cup B) \cap (A_{k+1} \cup B)$

Conclusion: Through direct proof, we proved $P(k) \to P(k+1)$

3. Prove that if x^3 is irrational, then x is irrational. Show all your work clearly state your conclusion.

P : x^3 is irrat

Q : x is irrat

Prove $P \to Q$

Proof by Contradiction

Assume $P, \neg Q$; find contradiction

x^3 is irrat x is rat

x = rat = a/b $a, b \in \mathbb{Z}$, lowest terms, $b \neq 0$

$\cancel{x^3}$

$(x)(x)(x) = (\text{rat})(\text{rat})(\text{rat})$

$x^3 = (a/b)(a/b)(a/b)$

$= \frac{a^3}{b^3}$ ← int

$x^3 = \cancel{\frac{(\text{int})^3}{b}} \; \frac{(\text{int})^3}{(\text{int})^3} = \frac{\text{int}}{\text{int}} = \text{rat}$

Contradicts P (x^3 = irrat)

$\boxed{\text{Found contradiction when assuming } P, \neg Q\text{; so Proof by contradiction holds, } P \to Q}$

$P(n)$

4. Prove that $3^n < n!$ if n is an integer greater than 6. Show all your work clearly state your conclusion.

Induction

Note, $n > 6$ so Base is $P(7)$

Base Step

$P(7)$

$3^7 < 7!$

$2187 < 5040$

Inductive Step

Assume $P(k)$: $3^k < k!$

Prove $P(k) \to P(k+1)$

Direct

$3^k < k!$

$P(k)$ $\underbrace{3 \cdot 3 \cdot 3 \cdots \cdot 3}_{k \text{ times}} < 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (k-1) \cdot k$

$P(k+1)$ $(\overbrace{3 \cdot 3 \cdot 3 \cdots \cdot 3}^{k \text{ times}}) \cdot 3 < 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (k-1) \cdot k (k+1)$

$3^k \cdot 3 < k! \cdot (k+1)$

We already know that $3^k < k!$ (from $P(k)$)

Since $3 < (k+1)$ (the multiples on each side),

$3^k \cdot 3 < k! \cdot (k+1)$ (Note: $k > 6$)

This proves the inductive step and therefore the proof by induction holds.

6 Extra Credit

Show that $4^{n+1} + 5^{2n-1}$ is always evenly divisible by **21** when $n > 0$.

$P(n)$: $4^{n+1} + 5^{2n-1} \% 21 = 0$

Induction

Base Step:

$P(1)$: $4^{1+1} + 5^{2(1)-1} = 4^2 + 5^1 = 16 + 5 = 21$ ✓

Inductive Step:

Assume $P(k)$: $4^{k+1} + 5^{2k-1} \% 21 = 0$

Prove $P(k) \rightarrow P(k+1)$

$P(k+1)$: $4^{(k+1)+1} + 5^{2(k+1)-1}$

$4^{k+1}4 + 5^{2k+1}$

$4^{k+1}4 + 5^{2k-1}5^2$

$4^{k+1}4 + 5^{2k-1}(21+4)$

$4^{k+1}4 + 5^{2k-1}(21) + 5^{2k-1}(4)$

$4^{k+1}4 + 5^{2k-1}(21) + 5^{2k-1}(4)$

~~4(4^k~~

$4(\underbrace{4^{k+1} + 5^{2k-1}}) + \underline{21(5^{2k-1})}$

divisible by 21 by definition

4 · (Divisible by 21 because of P(k))

Divisible by 21 by definition.

4 · divisible + divisible

divisible + divisible

divisible ✓