Hierarchical modeling of data gathered with (single-stage) cluster sampling: an example of how non-Bayesian estimators can produce rubbish results.

I'm studying quality of hospital care in California in 2011; (single-stage cluster sample).

to do this I take a random sample of I hospitals (in CA in 2011) and a random sample of J patients from the chosen hospitals (should really have the flexibility to take $J_i$ patients in hospital $i$, but let's look here at $J_i = J$ $\forall i$), and I measure a (AQIC) real-valued quality-of-care score $y_{ij}$ for patient $j$ in hospital $i$. To really understand what's going on I should also...
measure hospital-level and patient-level predictor variables (covariates) that help to explain QoC differences but even if we had these variables the following variance-component model would be a good starting point:

\[ Y_{ij} = \mu + \alpha_i + \epsilon_{ij} \]

\[ (i = 1, \ldots, I) \text{ hospitals} \quad i.i.d. \quad N(0, \sigma_H^2) \]

\[ (j = 1, \ldots, J_i) \text{ patients} \quad i.i.d. \quad N(0, \sigma_p^2) \]

The complete parameter vector in this model is \((\mu, \sigma_H^2, \sigma_p^2)\).

Typically, interest focuses on the grand mean \(\mu\), the variance components \(\sigma_H^2\) and \(\sigma_p^2\), and functions of \(\sigma_H^2\) and \(\sigma_p^2\) such as \(\frac{\sigma_H^2}{\sigma_H^2 + \sigma_p^2}\).

\(\mu\) is the population mean, quality of care score in CA in 2011, \(\alpha_i\) is the deviation of mean quality at hospital \(i\) from the grand mean, \(\epsilon_{ij}\) is the deviation of
quality for patient \( j \) from the mean quality at hospital \( i \); \( \sigma^2_H \) quantifies variations (between hospitals) from hospital to hospital in mean quality; \( \sigma^2_p \) quantifies variations within hospitals (between patients) in patient-level quality.

\[ V_{rs}(y_{ij}) = V_{rs}(\mu + a_i + q_{ij}) = \sigma^2_H + \sigma^2_p \]

is the total variance in quality, and

\[ \left( \frac{\sigma^2_H}{\sigma^2_H + \sigma^2_p} \right) \]

is the proportion of the total variance "accounted for" by variations in mean hospital quality.

Following Fisher's a standard frequentist analysis of this model is based on the analysis of variance table on the next page; you're supposed to compute sums of squares, degrees of freedom, and mean squares.
<table>
<thead>
<tr>
<th>Source</th>
<th>Sums of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>between hospitals</td>
<td>$SS_H = \frac{1}{J} \sum \limits_{i=1}^{J} (\bar{y}_i - \bar{y})^2$</td>
<td>$(I-1)$</td>
<td>$MS_H = \frac{SS_H}{I-1}$</td>
</tr>
<tr>
<td>within hospital</td>
<td>$SS_P = \frac{1}{I} \sum \limits_{i=1}^{I} \sum \limits_{j=1}^{J} (y_{ij} - \bar{y}_i)^2$</td>
<td>$IJ - I$</td>
<td>$MS_P = \frac{SS_P}{I(J-1)}$</td>
</tr>
<tr>
<td>total</td>
<td>$SS_T = \sum \limits_{i=1}^{I} \sum \limits_{j=1}^{J} (y_{ij} - \bar{y})^2$</td>
<td>$IJ - 1$</td>
<td></td>
</tr>
</tbody>
</table>

Here the dot subscripts denote averaging:

$\bar{y}_i = \frac{1}{J} \sum \limits_{j=1}^{J} y_{ij}$ and $\bar{y} = \frac{1}{IJ} \sum \limits_{i=1}^{I} \sum \limits_{j=1}^{J} y_{ij}$

Now focus on the various components; you can show the following things:

$$E_{RS}(MS_P) = \sigma^2_p \quad \text{and} \quad E_{RS}(MS_H) = J \sigma^2_H + \sigma^2_p$$

Hence, you would adopt a kind of method-of-moments attitude and equate...
\[
\begin{align*}
\left\{ \begin{array}{l}
MS_P = \hat{\sigma}_p^2 \\
MS_H = \frac{1}{J} \hat{\sigma}_H^2 + \hat{\sigma}_P^2
\end{array} \right\} \text{ and solve to get } \\
\left\{ \begin{array}{l}
\hat{\sigma}_P^2 = MS_P \\
\frac{\hat{\sigma}_H^2}{\sigma_H^2} = \frac{MS_H - MS_P}{J}
\end{array} \right\}
\end{align*}
\]

There are definitely unbiased estimates of the variance component, but \( \hat{\sigma}_H^2 \) is absurd because it can easily go negative.

This doesn't bother Schef\'fe (1959) (pp. 228-229), but he reveals with his attitude that he\'s never done any serious applied work.

He goes on to derive the following confidence interval for \( \theta = \frac{\hat{\sigma}_H^2}{\hat{\sigma}_P^2} \), which is related monotonically to \( \frac{\hat{\sigma}_H^2}{\hat{\sigma}_H^2 + \hat{\sigma}_P^2} \):

\[
\frac{1}{1 + \frac{\hat{\sigma}_P^2}{\hat{\sigma}_H^2}} = \frac{1}{1 + \frac{1}{\theta}} = \frac{\theta}{\theta + 1}
\]
\[
\frac{1}{J} \left( \frac{MS_H/MS_p}{F^{-1}(1-\frac{\alpha}{2}, I-1, I(J-1))} - 1 \right) < \theta < \frac{1}{J} \left( \frac{MS_H/MS_p}{F^{-1}(\frac{\alpha}{2}, I-1, I(J-1))} - 1 \right)
\]

F curve with \((I-1)\) and \(I(J-1)\) degrees of freedom.

Not only is it possible for this interval to have a negative left endpoint; the entire interval for \((\frac{\sigma_H^2}{\sigma_p^2})\) can be negative! This also does not bother Scheffé (1959, pp. 229-231). His position on this issue shows that he values unbiasedness more highly than sanity. (Reader winbury demo in class)