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ADER: Arbitrary High Order Godunov Approach

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This paper concerns the construction of non-oscillatory schemes of very high order of accuracy in space and time, to solve non-linear hyperbolic conservation laws. The schemes result from extending the ADER approach, which is related to the ENO/WENO methodology. Our schemes are conservative, one-step, explicit and fully discrete, requiring only the computation of the inter-cell fluxes to advance the solution by a full time step; the schemes have optimal stability condition. To compute the intercell flux in one space dimension we solve a generalised Riemann problem by reducing it to the solution a sequence of m conventional Riemann problems for the kth spatial derivatives of the solution, with $k = 0, 1, \ldots, m-1$, where m is arbitrary and is the order of the accuracy of the resulting scheme. We provide numerical examples using schemes of up to fifth order of accuracy in both time and space.

KEY WORDS: ADER; essentialy non-oscillatory; Godunov; generalised Riemann problem.

1. INTRODUCTION

In designing numerical schemes of very high-order of accuracy for solving hyperbolic conservation laws one faces at least three major difficulties. One of them concerns the preservation of high accuracy in both space and time for multidimensional problems containing source terms. Another one concerns conservation; this is mandatory in the presence of shock waves. The other very important issue relates to the generation of spurious oscillations in the vicinity of strong gradients; according to Godunov's theorem [5] these are unavoidable by linear schemes of accuracy greater than one! Each one of these difficulties is in itself not easy to resolve; the simultaneous resolution of all three difficulties above represents a formidable task in the numerical analysis of hyperbolic conservation laws. At present, there are various approaches for constructing numerical schemes that attempt to overcome the above difficulties. State-of-art very high order methods (at

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least third order) for hyperbolic conservation laws include the class of ENO/WENO schemes [1, 6, and 7], Spectral Methods [3], the class of Compact Difference Methods [12] and Discontinuous Galerkin Finite Element Methods [4]. These schemes meet the requirement of very high-order spatial accuracy; matching time accuracy to space accuracy, however, remains an issue in all of the above approaches. As regards the ENO/WENO/MPWENO approach, the most accurate scheme reported so far uses 9th order spatial discretisation with Runge–Kutta methods for time evolution. To avoid spurious oscillations these Runge Kutta methods must be TVD. This leads to accuracy barriers: the accuracy of such methods cannot be higher than fifth. Moreover, fourth and fifth order methods are quite complicated and have reduced stability range. In most practical implementations reported, when the solution is not smooth, a third order TVD Runge–Kutta method has been used, e.g., [1]. Although increased order of spatial descretisation improves accuracy for some problems such schemes converge with third order only when the mesh is refined and thus should be regarded as third order schemes. For some applications, such as numerical simulation of compressible turbulence and wave propagation problems involving long-time evolution it would be beneficial to use schemes which converge with higher order both in time and space.

A recent approach for constructing schemes of very high order of accuracy is the ADER (Advection-Diffusion-Reaction) approach [10], which stems from the modified Generalised Riemann Problem (MGRP) scheme [14] which in turn is a simplification of the GRP scheme [2]. The approach has so far been applicable only to linear hyperbolic systems with constant coefficients in one and multiple space dimensions, see also [9]. The schemes are conservative, one-step, explicit and fully discrete, requiring only the computation of inter-cell fluxes to advance the solution by a full time step.

In this paper we extend the ADER approach to nonlinear hyperbolic systems. The designed finite-volume schemes are of arbitrary order of accuracy in *both time and space* and with *optimal* stability condition for all problems. The first order ADER scheme boils down to the original Godunov's scheme [5] and second and higher order schemes can be regarded as its higher order extensions. Our generalisation has some similarities with finite-volume ENO schemes [7] but differs in the way time accuracy is preserved as will be explained below.

The rest of the paper is organized as follows. In Section 2 we describe the ADER approach for hyperbolic systems. Numerical examples are given in Section 3 and conclusions are drawn in Section 4.

2. THE NUMERICAL SCHEME

We consider the hyperbolic system in conservation form given by

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0} \tag{1}$$

along with initial and boundary conditions. Here $\mathbf{Q}$ is the vector of unknown conservative variables and $\mathbf{F}(\mathbf{Q})$ is the physical flux.

Consider now a control volume in $x-t$ space $I_i \times [t^n, t^{n+1}]$, $I_i = [x_{i-1/2}, x_{i+1/2}]$, of dimensions $\Delta x = x_{i+1/2} - x_{i-1/2}$, $\Delta t = t^{n+1} - t^n$. Integrating (1) on $I_i \times [t^n, t^{n+1}]$ we obtain

$$\bar{\mathbf{Q}}(x_i, t^{n+1}) = \bar{\mathbf{Q}}(x_i, t^n) - \frac{\Delta t}{\Delta x}\left[\int_{t^n}^{t^{n+1}} \mathbf{F}(\mathbf{Q}(x_{i+1/2}, t))\, dt - \int_{t^n}^{t^{n+1}} \mathbf{F}(\mathbf{Q}(x_{i-1/2}, t))\, dt\right] \tag{2}$$

where $\bar{\mathbf{Q}}(x_i, t^n)$ is the sliding average of the solution in cell I_i at time t^n. We approximate (2) by the following conservative scheme:

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}(\hat{\mathbf{F}}_{i+1/2} - \hat{\mathbf{F}}_{i-1/2}) \tag{3}$$

where $\mathbf{Q}_i^n$ is a high-order approximation to the cell average $\bar{\mathbf{Q}}(x_i, t^n)$ and $\hat{\mathbf{F}}_{i+1/2}$ is the numerical flux. Note that although the scheme describes the evolution of cell averages the numerical flux involves point-wise values of the unknown variables.

The ADER approach consists of three steps: (i) reconstruction of point wise values from cell averages, (ii) solution of the generalised Riemann problem at the cell interface and (iii) evaluation of the intercell flux to be used in the conservative scheme (3). The point-wise values of the solution at $t = t^n$ are reconstructed from cell averages via high-order polynomials. Use of a fixed stencil for the reconstruction leads to linear ADER schemes (a scheme is called linear if it has constant coefficients when applied to a linear equation with constant coefficients). However, in accordance with Godunov's theorem [5], linear schemes of order of accuracy higher than one will produce spurious oscillations near discontinuities and steep gradients of the solution. To avoid oscillations the adaptive WENO reconstruction technique is used leading to nonlinear ADER schemes. In the rest of the paper we denote mth order ADER schemes as ADERm. By the order of accuracy we mean the convergence rate of the scheme when the mesh is refined with a fixed Courant number. ADER3 uses weighted parabolic reconstruction; ADER4 uses weighted cubic reconstruction and so on.

After the reconstruction step the conservative variables are represented as vectors $\mathbf{p}_i(x)$ of polynomials. At each cell interface we have the following generalised Riemann problem with the reconstruction polynomials of $(m-1)$th order for an mth order scheme:

$$\begin{aligned} &\text{PDE:} \quad \partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0} \\ &\text{IC:} \quad \mathbf{Q}(x, 0) = \begin{cases} \mathbf{Q}_L(x) = \mathbf{p}_i(x), & x < x_{i+1/2} \\ \mathbf{Q}_R(x) = \mathbf{p}_{i+1}(x), & x > x_{i+1/2} \end{cases} \end{aligned} \tag{4}$$

We find an approximate solution for the interface state $\mathbf{Q}(x_{i+1/2}, \tau)$, where τ is local time $\tau = t - t^n$, using a semi-analytical method by Toro and Titarev [11, 13]. This generalisation of the Riemann problem is twofold: (i) the governing equations include non-linear advection as well as reaction terms and (ii) the initial condition consists of two arbitrary but infinitely differentiable functions. The generalised Riemann problem with polynomials of m order is then denoted as GRP_m. The method gives the solution at $x = x_{i+1/2}$ at a time τ, assumed to be sufficiently small, in terms of solutions of a sequence of conventional Riemann problems for homogeneous advection equations.

In short, the approximate solution $\mathbf{Q}(x_{i+1/2}, \tau)$ can be evaluated as follows. First we write a Taylor expansion of the interface state in time

$$\mathbf{Q}(x_{i+1/2}, \tau) = \mathbf{Q}(x_{i+1/2}, 0^+) + \sum_{k=1}^{m-1} [\partial_t^{(k)} \mathbf{Q}(x_{i+1/2}, 0^+)] \frac{\tau^k}{k!} \tag{5}$$

where

$$\partial_t^{(k)} \mathbf{Q}(x, t) = \frac{\partial^k}{\partial t^k} \mathbf{Q}(x, t), \qquad 0^+ \equiv \lim_{\tau \to 0^+} \tau$$

The leading term $\mathbf{Q}(x_{i+1/2}, 0^+)$ accounts for the interaction of the boundary extrapolated values $\mathbf{Q}_L(x_{i+1/2})$ and $\mathbf{Q}_R(x_{i+1/2})$ and is the Godunov state of the conventional (piece-wise constant data) GRP_0

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0}$$
$$\mathbf{Q}(x, 0) = \begin{cases} \mathbf{Q}_L(x_{i+1/2}) & \text{if} \quad x < x_{i+1/2} \\ \mathbf{Q}_R(x_{i+1/2}) & \text{if} \quad x > x_{i+1/2} \end{cases} \tag{6}$$

A key ingredient is the availability of an exact or approximate Riemann solver for the conventional Riemann problem GRP_0 to provide the first term in the expansion. Note that since all other terms are higher order corrections to the Godunov state the scheme is indeed an arbitrary high order extension of the Godunov scheme.

Next, we replace all time derivatives by space derivatives using the governing system of equation by means of the Lax–Wendroff procedure [8]. For example, for the scalar linear equation $q_t + \lambda q_x = 0$, with constant λ, we have

$$\partial_t^{(k)} q(x, t) = (-\lambda)^k \, \partial_x^{(k)} q(x, t)$$

and the expansion becomes

$$q(x_{i+1/2}, \tau) = q(x_{i+1/2}, 0^+) + \sum_{k=1}^{m-1} [\partial_x^{(k)} q(x_{i+1/2}, 0^+)] \frac{(-\lambda\tau)^k}{k!} \tag{7}$$

The Lax–Wendroff procedure for the Euler equations is described in [7]. One may use some algebraic manipulator for this.

Next, it can be shown [11, 13] that space derivatives $\mathbf{Q}^{(k)} \equiv \partial_x^{(k)} \mathbf{Q}$ of the solution at $(x - x_{i+1/2})/\tau = 0$ can be evaluated as the Godunov states of the the following linearised GRP_0:

$$\partial_t \mathbf{Q}^{(k)} + \mathbf{A}(\mathbf{Q}(x_{i+1/2}, 0^+))\, \partial_x \mathbf{Q}^{(k)} = \mathbf{0}$$

$$\mathbf{Q}^{(k)}(x, 0) = \begin{cases} \partial_x^{(k)} \mathbf{Q}_L(x_{i+1/2}), & x < x_{i+1/2} \\ \partial_x^{(k)} \mathbf{Q}_R(x_{i+1/2}), & x > x_{i+1/2} \end{cases} \tag{8}$$

The initial condition for (8) is found by differentiating the given WENO reconstruction polynomial with respect to x. In general, optimal weights for derivatives do not always exist; for example if we use weighted parabolic reconstruction there are no optimal weights for the first derivative. Because of this, we use the same weights and smoothness indicators for the function and for all derivatives. As a result, the ADERm scheme is of mth order in space (and time) despite the fact that point-wise values of the solution are reconstructed with $(2m-1)$ order of accuracy.

Note that the coefficient matrix $\mathbf{A}$ is the same for all $\mathbf{Q}^{(k)}$ and has to be evaluated only once. Finally, having found all space derivatives we form the Taylor expansion (5):

$$\mathbf{Q}(x_{i+1/2}, \tau) = A_0 + A_1 \tau + A_2 \tau^2 + \cdots + A_{m-1} \tau^{m-1}, \qquad 0 \leqslant \tau \leqslant \Delta t$$

which approximates the interface state for $0 \leqslant \tau \leqslant \Delta t$ to mth order of accuracy. To evaluate the numerical flux we use an appropriate Gaussian rule:

$$\hat{\mathbf{F}}_{i+1/2} = \sum_{\alpha=0}^{K_\alpha} \mathbf{F}(\mathbf{Q}(x_{i+1/2}, \gamma_\alpha\, \Delta t))\, \omega_\alpha \tag{9}$$

where γ_j and ω_j are properly scaled nodes and weights of the rule and K_α is the number of nodes.

A very important issue is stability of the numerical method just described. Linear schemes applied to the linear advection equation with constant coefficient have the optimal stability condition CFL $\leqslant 1$ [10], where CFL denotes the (maximum) Courant number. Numerical experiments indicate that the approach has the same stability condition for nonlinear scalar equations and systems as well. When solving nonlinear equations in practice, however, smaller CFL numbers are used. For the examples presented here we use CFL $= 0.95$ throughout.

Remark 1. We note that our generalisation of the ADER approach has some similarities with finite volume ENO schemes developed in [7]. However, the key difference lies in how time accuracy is preserved. In

the ENO scheme one extrapolates the boundary values of conservative variables forward in time via the Lax–Wendroff procedure in the nodes of the Gaussian rule used to integrate the flux in time for $t^n \leqslant t \leqslant t^{n+1}$. Then, Riemann problems with these values as initial conditions are solved, and the flux is evaluated at the nodes. These values of the flux are summed with the Gaussian weights to get the approximate expression for the flux. In contrast, we build the *state* expansion in time using the concept of generalised Riemann problem and then integrate the flux using this expansion. Another major difference is that the ADER approach extends directly to handle source terms, which, in a very natural way, enter the flux computation via the solution of the inhomogeneous generalised Riemann problem; see [11] for numerical implementations for the scalar case.

Remark 2. An important issue is efficiency of the schemes. For the linear equation with constant coefficients ADER schemes are 2–3 times faster than corresponding ENO/WENO schemes when the same CFL number is used [10]. For the 1D Euler equations, our provisional results show that the ADER3 scheme is about 50% faster than WENO5 scheme. This is due to the fact that in the WENO scheme one has to evaluate characteristic variables and smoothness indicators in each reconstruction step and then solve the nonlinear Riemann problem at the cell interface; this is done *three times* for the third order Runge–Kutta method. In practical applications finite-difference WENO schemes usually need rather small CFL numbers ($0.2 \cdots 0.6$). Also, it is worth mentioning that in the monotonicity constraint used in the MPWENO scheme one requires CFL $\leqslant 0.2$, although the scheme appears to be non-oscillatory for CFL numbers of $0.4 \cdots 0.6$, depending on the problem of course. In contrast, ADER schemes can use CFL numbers close to the optimal for all problems [11]. The range of CFL numbers that a given scheme can use in practical computations must also be a factor in discussing efficiency. In 2D, finite-difference schemes are in general faster than finite-volume schemes. However, we expect the difference to be smaller for ADER than for WENO schemes, because in the former the reconstruction is done only once.

3. NUMERICAL EXAMPLES

Here we study numerically the convergence properties of ADER schemes and compare them with the state-of-art WENO5 scheme, which uses a fifth-order accurate weighted parabolic spatial reconstruction and a third order TVD Runge–Kutta time stepping scheme. WENO5 is fifth order in space and third order in time and consequently converges with third-order of accuracy.

Example 1. We solve $\partial_t q + \partial_x q = 0$ with the initial condition $q_0(x) = \sin^4(\pi x)$ defined on $[-1, 1]$. Periodic boundary conditions are used. The error is measured at time $t = 1$. Table I shows convergence rates and errors

Table I. Linear Advection. IC: $q_0(x) = \sin^4(\pi x)$. Output Time $t = 1$, CFL = 0.95, N Is the Number of Cells

Method	N	L_∞ error	L_∞ order	L_1 error	L_1 order
WENO5	20	1.37×10^{-1}		1.32×10^{-1}	
	40	2.29×10^{-2}	2.58	2.00×10^{-2}	2.72
	80	2.50×10^{-3}	3.19	2.39×10^{-3}	3.06
	160	2.82×10^{-4}	3.15	2.78×10^{-4}	3.11
ADER3	20	1.97×10^{-2}		1.83×10^{-2}	
	40	2.65×10^{-3}	2.89	1.38×10^{-3}	3.72
	80	3.42×10^{-4}	2.96	1.26×10^{-4}	3.46
	160	2.45×10^{-5}	3.80	1.49×10^{-5}	3.08
ADER5	20	2.91×10^{-3}		3.22×10^{-3}	
	40	1.23×10^{-4}	4.56	5.53×10^{-5}	5.86
	80	1.96×10^{-6}	5.97	6.78×10^{-7}	6.35
	160	1.18×10^{-8}	7.37	1.28×10^{-8}	5.73

for different schemes. We observe that all ADER schemes reach the designed order of accuracy. As expected, the error of WENO5 decreases with third order only, despite the fact that it has fifth order spatial accuracy. We note that ADER3 is more accurate than WENO5 and ADER5 achieves substantially better accuracy than both third order schemes, as expected.

Example 2. We solve the inviscid Burgers' equation $\partial_t q + \partial_x(\frac{1}{2} q^2) = 0$ with the initial condition $q_0(x) = 0.25 + 0.5 \sin(\pi x)$, defined on $[-1, 1]$. Periodic boundary conditions are used. The error is measured at the output time $t = 0.2$. Table II shows the numerical results. We observe that the ADER schemes reach the designed order of accuracy. Again ADER5 is much more accurate than both third order schemes; the gap in accuracy rapidly increases as the mesh is refined.

Table II. Burgers' Equation, IC: $q_0(x) = 0.25 + 0.5 \sin(\pi x)$. Output Time $t = 0.2$, CFL = 0.95, N Is Number of Cells

Method	N	L_∞ error	L_∞ order	L_1 error	L_1 order
WENO5	20	1.25×10^{-3}		6.58×10^{-4}	
	40	2.51×10^{-4}	2.31	9.74×10^{-5}	2.76
	80	3.12×10^{-5}	3.01	1.23×10^{-5}	2.99
	160	3.83×10^{-6}	3.03	1.55×10^{-6}	2.98
ADER3	20	1.02×10^{-3}		6.63×10^{-4}	
	40	2.65×10^{-4}	1.95	1.16×10^{-4}	2.52
	80	3.45×10^{-5}	2.94	1.47×10^{-5}	2.98
	160	4.41×10^{-6}	2.97	1.86×10^{-6}	2.98
ADER5	20	1.04×10^{-4}		5.58×10^{-5}	
	40	6.00×10^{-6}	4.12	2.09×10^{-6}	4.74
	80	1.97×10^{-7}	4.93	6.15×10^{-8}	5.09
	160	6.05×10^{-9}	5.02	1.88×10^{-9}	5.03

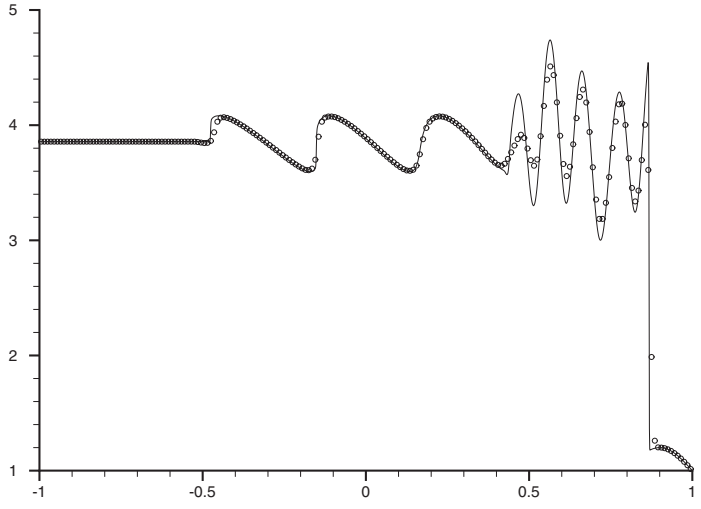

Fig. 1. Shock-turbulence interaction test problem. Density profile computed by WENO5 with 200 cells and CFL = 0.95.

Example 3. Shock turbulence interaction problem [1]. We now assess the performance of the ADER schemes for the 1D Euler equations for a γ-law gas with $\gamma = 1.4$ in all the results shown here. The initial condition defined on $[-1, 1]$ is

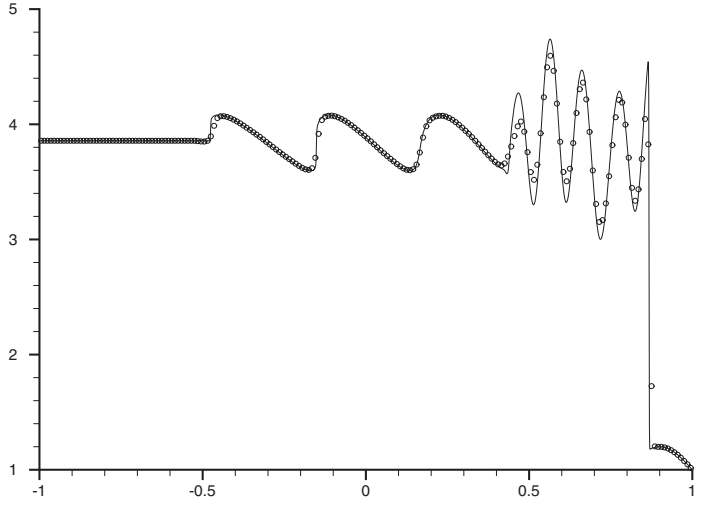

Fig. 2. Shock-turbulence interaction test problem. Density profile computed by ADER3 with 200 cells and CFL = 0.95.

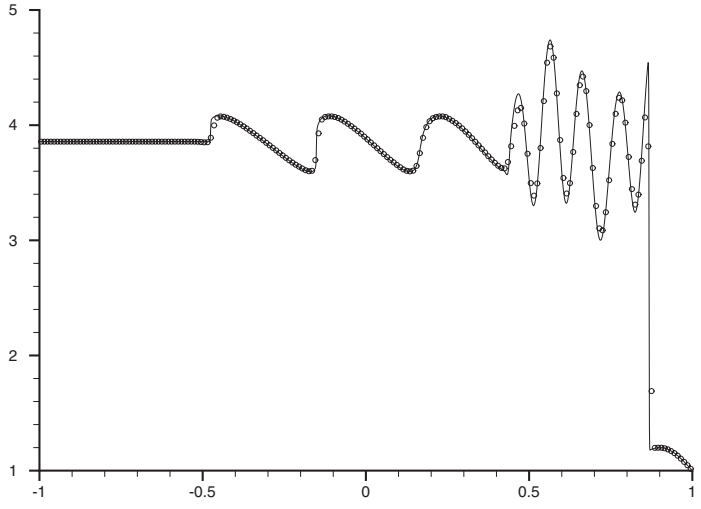

Fig. 3. Shock-turbulence interaction test problem. Density profile computed by ADER4 with 200 cells and CFL = 0.95.

$$(\rho, u, p) = \begin{cases} (3.857143, 2.629369, 10.333333), & x < -0.8 \\ (1 + 0.2 \sin 5\pi x, 0.0, 1.), & x > -0.8 \end{cases}$$

Output time is $t = 0.47$ and a mesh of 200 cells is used. A reference solution is computed on a fine mesh with 2000 cells and is shown as a solid line in the graphs. Figs. 1–3 show density profiles computed by WENO5, ADER3 and ADER4. We observe that ADER3 is much more accurate than WENO5 whereas ADER4 outperforms both of them and is comparable with MPWENO (see [1]).

4. CONCLUSIONS

An extension of the ADER approach to nonlinear hyperbolic systems was presented. It uses the solution technique for the generalised Riemann problem for non-linear systems with source terms developed by Toro and Titarev [11, 13]. Unlike the WENO schemes the developed ADER schemes are of unbounded accuracy in both space and time, and have optimal stability condition for explicit schemes. The numerical results indicate that for the 1D problems considered the schemes compare very favourably with state-of-art WENO schemes. Preliminary results for the linearised Euler equations in 2D appear to be equal or superior to ENO/WENO schemes [9]. The extension to nonlinear hyperbolic systems in 2D is the subject of ongoing research.

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