AMS260 Homework 3

Theory Problems (MANDATORY)

Problem 1 Consider the Lax-Friedrichs (LF) method for solving the scalar advection $u_t + f(u)_x = 0$ with $f(u) = au$, where $a > 0$ or $a < 0$,

$$U_i^{n+1} = \frac{1}{2}\Big(U_{i+1}^n + U_{i-1}^n\Big) - \frac{\Delta t}{2\Delta x}\Big(f(U_{i+1}^n) - f(U_{i-1}^n)\Big). \tag{1}$$

(a) Show that the LF method is consistent and stable for $|C_a| \le 1$.

(b) Show that the LF method is $\mathcal{O}(\Delta t + \Delta x^2)$.

(c) Rewrite the LF method in the conservative form,

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\Big(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n\Big), \tag{2}$$

that is to say, please find expressions for $\hat{f}_{i\pm 1/2}^n$ as functions of U_k^n and the original flux $f(U_k^n)$, $k = -1, 0, 1$.

Problem 2 Consider the Lax-Wendroff (LW) method for solving the scalar advection $u_t + au_x = 0$ with $a > 0$ and $f(u) = au$,

$$U_i^{n+1} = U_i^n - \frac{C_a}{2}\Big(U_{i+1}^n - U_{i-1}^n\Big) + \frac{C_a^2}{2}\Big(U_{i+1}^n - 2U_i^n + U_{i-1}^n\Big). \tag{3}$$

(a) Show that the LW method is consistent and stable if $|C_a| \le 1$.

(b) Show that the LW method is $\mathcal{O}(\Delta t^2 + \Delta x^2)$.

Problem 3 Use the von Neumann analysis of the 1D advection using forward in time forward in space (FTFS)

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{\Delta x}\Big(U_{j+1}^n - U_j^n\Big) \tag{4}$$

to show that FTFS is unstable if $a > 0$ and stable if $a < 0$.

Problem 4 Show that a forward in time centered in space scheme (FTCS) for 1D advection with $a > 0$

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{2\Delta x}\Big(U_{j+1}^n - U_{j-1}^n\Big) \tag{5}$$

is unconditionally unstable (i.e., stable for any choices of $\Delta t > 0$).

Problem 5 Show that an implicit scheme of backward in time centered in space (BTCS)

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{2\Delta x}\Big(U_{j+1}^{n+1} - U_{j-1}^{n+1}\Big) \tag{6}$$

is unconditionally stable.

Problem 6 A typical linear advection equation $u_t + au_x = 0$ may be written in the finite difference form

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = -a\Big[\theta \frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2\Delta x} + (1-\theta)\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}\Big] \tag{7}$$

(a) Use von Neumann stability analysis to obtain three amplification factors for $\theta = 0$, $1/2$ and 1, respectively.

become

$$A = \begin{cases} 1 + IC_a \sin(k\Delta x) \text{ if } \theta = 0, \\ \frac{1-\frac{1}{2}IC_a\sin(k\Delta x)}{1+\frac{1}{2}IC_a\sin(k\Delta x)} \text{ if } \theta = \frac{1}{2}, \\ \frac{1}{1+IC_a\sin(k\Delta x)} \text{ if } \theta = 1, \end{cases} \tag{8}$$

where $C_a = a\Delta t/\Delta x$ is a CFL number, k is a wave number and $I = \sqrt{-1}$.

(b) Discuss the stability of the three cases, $\theta = 0$, $1/2$ and 1, based on the amplification factors you have found in (a).

Problem 7 Consider the equation

$$u_t + au_x = \beta u. \tag{9}$$

(a) Show that the method for this PDE

$$U_i^{n+1} = U_i^n - \frac{a\Delta t}{\Delta x}\Big(U_i^n - U_{i-1}^n\Big) + \Delta t \beta U_i^n \tag{10}$$

is first-order accurate for this equation by computing the local truncation error.

(b) Is the method consistent?

Coding Problems (OPTIONAL)

Use Fortran 90 or C to implement the following schemes. To practice modular programming in Fortran 90, please follow the link provided separately in the homework 3 webpage. Please use the sample Matlab code available on the course Git repository.

Note: Undergrad students have options to use Matlab or Python for the coding problems. So is for the final coding project.

Problem 8 Implement the LF method in Eqn. (1) to numerically solve the sinusoidal advection problem

$$u_t + au_x = 0, \quad a = 1, \tag{11}$$

with an IC: $u(x, 0) = \sin(x)$, on $x \in [0, 2\pi]$. Use the periodic boundary conditions (BC). Run your code on two different grid resolutions of $N = 32, 128$ with $C_a = 0.8$. Please show your plots at $t = t_{cycle1}$ at all two grid resolutions, where t_{cycle1} is the time the sinusoidal wave returns to the initial position (Hint: You can easily find t_{cycle1} analytically first). Describe your findings and compare the LF results with the first-order upwind method provided in the Matlab code.

Problem 9 Repeat Problem 8 using the LW method in Eqn. (3).