AMS260 Homework 2

Problem 1 Consider Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{1}$$

(a) By multiplying the equation by $2u$, show that you can derive a new conservation law for u^2. What is the new flux function?

(b) Show that the original Burgers' equation and the new derived equation have different weak solutions (Hint: It suffices to show that there exist two different shock speeds from the two equations for the Riemann problem with $u_l > u_r$.).

Problem 2 Solve Burgers' equation on $\mathbb{R}$ for small enough $t \leq t_b$ that allows the exact piecewise-linear weak solution with the following initial conditions:

$$u(x,0) = \begin{cases} 2 & \text{if } |x| < 1/2 \\ -1 & \text{if } |x| > 1/2 \end{cases} \tag{2}$$

Find the time t_b when the tail of the rarefaction and the shock wave first intersect each other. Draw a wave diagram for the weak solution in the x-t plane.

Problem 3 Consider the scalar conservation law $u_t + (\frac{e^u}{2})_x = 0$ with initial data $u(x,0) = u_0(x)$:

$$u_0(x) = \begin{cases} 2 \text{ if } -1 < x < 1, \\ 0 \text{ otherwise }. \end{cases} \tag{3}$$

(a) Sketch the characteristics and shock paths in the x-t plane. Please clearly identify the exact solution in each compression and rarefaction region in the x-t plane. Use $e^2 \approx 7.38$.

(b) Find $t = t_b$ at which the shock and the expansion fan begin to cross.

Problem 4 Let $u(x,t)$ be defined for $(x,t) \in \mathbb{R}^2$ by

$$u(x,t) = \begin{cases} 1 \text{ for } x < t/2 \\ 0 \text{ for } x > t/2. \end{cases} \tag{4}$$

(a) By using the definition of a weak solution, show that u is a weak solution of $u_t + uu_x = 0$. Please assume your test functions $\phi(x,t)$ is continuously differentiable with compact support, i.e., $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$.

(b) Show that u satisfies the integral form

$$\frac{d}{dt}\int_a^b u(x,t)dx = F(a,t) - F(b,t) \tag{5}$$

of the conservation law when $F(u) = \frac{u^2}{2}$. (Hint: Consider three cases: (i) $t/2 < a < b$, (ii) $a < t/2 < b$, and (iii) $a < b < t/2$.)