AMS260 Homework 1

Problem 1 In Chapter 1, we derived four different equations assuming four different models. Show that all four approaches discussed in (F1)-(F4) for the continuity equation are in fact all equivalent mathematically. That is, one of them can be obtained from any of the others. (Hint: You can show that there are equivalent relationships in circle: (F1) $\Rightarrow$ (F2) $\Rightarrow$ (F4) $\Rightarrow$ (F3) $\Rightarrow$ (F1)).

Problem 2 (You don't need to submit your work for Problem 2) Study the two MATLAB codes for 1D advection and diffusion, available at the course git repo (https://bitbucket.org/dongwook159/ams260-winter-2017). Also, please study **Homework 1** and **Homework 2** problems on page 22 and page 23 in the lecture note.

Problem 3 Consider the vanishing viscosity equation,

$$u_t + au_x = \epsilon u_{xx}. \tag{1}$$

Use a change of variables to follow the characteristics (i.e., $\xi = x + at$ and $\tau = t$) and set

$$v^\epsilon(x,t) = u^\epsilon(x+at,t) = u^\epsilon(\xi,\tau) \tag{2}$$

(a) Assume u^ϵ is a solution to Eq. (1) and show that v^ϵ satisfies the heat equation

$$v^\epsilon_t(x,t) = \epsilon v^\epsilon_{xx}(x,t). \tag{3}$$

Note that we have converted the advection-diffusion equation Eq. (1) to the pure diffusion equation Eq. (3). Now, using the well-known solution to the diffusion equation to solve for $v^\epsilon(x,t)$ (see Hint 1),

(b) Show that we have (this should be very trivial)

$$u^\epsilon(x,t) = v^\epsilon(x-at,t). \tag{4}$$

(c) And moreover, show that (use Hint 2)

$$\lim_{\epsilon\to 0} u^\epsilon(x,t) = \lim_{\epsilon\to 0} u^\epsilon(x,t) = u_0(x-at). \tag{5}$$

Hint 1: For the diffusion equation Eq. (3), we can always find the classical solution of the PDE using Green's functions:

$$v^\epsilon(x,t) = \frac{1}{\sqrt{4\pi\epsilon t}}\int_R e^{-\frac{(x-y)^2}{4\epsilon t}} v^\epsilon(y,0)dy, \tag{6}$$

where $v^\epsilon(y,0) = u^\epsilon(y,0) = u^\epsilon_0(y)$.

Hint 2: Let $g(x)$ be a bounded function and is continuous at $x = 0$. Let

$$\gamma_r(x) = \sqrt{\frac{r}{\pi}}e^{-rx^2}. \tag{7}$$

Then

$$\lim_{r\to\infty} \int_{\mathbb{R}} \gamma_r(x-y)g(x)dx = g(y). \tag{8}$$

Note that $\gamma_r(x)$ is Gaussian and has the following properties:
(a) $\gamma_r(x) \geq 0$,
(b) $\lim_{r\to\infty} \gamma_r(x) = 0$ if $x \neq 0$; $\lim_{r\to\infty} \gamma_r(x) = 0$ if $x = 0$,
(c) $\int_{\mathbb{R}} \gamma_r(x)dx = \sqrt{\frac{r}{\pi}} \int_{\mathbb{R}} e^{-rx^2} dx = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} dy = 1$.

Problem 4 Given a smooth initial data $u_0(\xi)$ for Burgers' equation with its slope $u_0'(\xi) < 0$ at some point ξ_0. Show that the wave break time t_b is written as

$$t_b = \frac{-1}{u_0'(\xi_0)}. \tag{9}$$

(Hint: Prove that $u_x \to \infty$ when the wave breaks at $t = t_b$. Use $u(x,t) = u_0(\xi)$ and $x - u_0(\xi)t = \xi$ to derive u_x.)

Problem 5 Try modifying the 1D advection MATLAB code to solve the Burgers' equation by replacing a with u_i. You can keep the setups unchanged including boundary condition, grid resolution, CFL number, etc. Does your solution stable or unstable? If unstable, can you explain why? Can you suggest how to resolve the issue? (Yet, you haven't learned enough to provide a correct answer, but please try your best to give a good resolution to the issue.)