Chapter 9. FVM for the Euler Eqns.

→ We will study the system of eqns, e.g. the Euler eqns.

→ The Euler eqns are <u>non-linear</u>, hence one question naturally follows;

Q: What we learned about linear eqns are useless then?

A: No!

→ We will fully make use of our knowlege on linear eqns via the following steps;

(i) local linearizations of the nonlinear flux Jacobian matrix $\frac{\partial F}{\partial U}$,

(ii) diagonalize the linearized system of eqns,

(iii) decouple the system into a group of separate piece of eqns.

(iv) individual eqns are then <u>linear scalar eqns.</u>

→ The goal in this chapter is to learn num. techniques for the Euler eqns.

→ The nature of nonlinearity in the Euler eqns provides <u>complicated</u> <u>wave</u> <u>structures</u> that are <u>multiple</u>.

→ The characteristic information becomes much richer than the simple single-wave form in the scalar eqn case.

→ The RP is consist of multiple jumps across each characteristics.

→ The num. flux should account for these jump conditions over multiple char. waves

→ More sophisticated num. flux formulation than the scalar eqn case (e.g., scalar advection, Burgers)

→ We will see this in Roe & HLL, both of which are <u>approximate</u> Riemann solvers.

→ On the other hand, one can also obtain "exact" Riemann solvers, for both hydro & MHD.

→ <u>Exact</u> : <u>Iterative</u>

→ Approximate Riemann solvers (Good enough in practice)

LLF (Rusanov)

HLL

HLLC

HLLD

Roe, Two-shock, Two-Rarefaction,

Osher's, hybrid, etc. (<u>1D based</u>)

→ The choice of Riemann solvers affects

{ soln. accuracy, &
soln. <u>stability</u>. ☺ <u>important</u>

→ Also, there are multi-D Riemann solvers.

Linear Hyperbolic Systems

→ Consider the linear system of conservation laws;

$$(*) \begin{cases} U_t + (AU)_x = 0 \\ U(x,0) = U_0(x), \end{cases}$$

$U : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^m$, $A \in \mathbb{R}^{m \times m}$

eg $(x,t) \mapsto \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{pmatrix}$ $m=5$ $\quad \overset{\text{def}}{=} \dfrac{\partial F(U)}{\partial U}$.

→ $\begin{cases} U : \text{conservative quantity}, \\ F(U) : \text{flux vector} \end{cases}$, given by $\underbrace{F(U) = AU}$

resembles with scalar advection

Def The system of conservation law $(*)$ is called <u>hyperbolic</u> if A is diagonalizable with real eig. values

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m, \quad \text{so that}$$

we can decompose

$$A = R \Lambda R^{-1},$$

$$\begin{cases} \Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m) \text{ ; diagonal matrix} \\ R = [r_1 | r_2 | \cdots | r_m] \text{ ; the matrix whose columns are right eig. vectors } r_k. \end{cases}$$

$R^{-1} = L$; the matrix whose rows are left eig. vectors l_k.

→ Note $\cancel{r_i \cdot l_j = \delta_{ij}}$. $l_i \cdot r_j = \delta_{ij}$

→ Note $AR = R\Lambda$, i.e,

$$A r_k = \lambda_k r_k, \quad k = 1, 2, \cdots, m.$$

Def The system is called <u>strictly hyperbolic</u> if

$$\lambda_1 \lneq \lambda_2 \lneq \cdots \lneq \lambda_k.$$

<u>Rmk</u> We will assume to have the strictly hyperbolic system, especially when considering 1D.

$\boxed{\text{Diagonalization of the coupled linear system} \rightarrow \text{Decoupled system of linear eqns.}}$

→ Using $A = R \Lambda R^{-1}$, we rewrite

$U_t + A U_x = 0$ as
← linear system

$$\Rightarrow \quad R^{-1} U_t + \Lambda R^{-1} U_x = 0$$

⇒ Since R^{-1} is constant, if we let $W = R^{-1} U = L U$, then we obtain

$$\boxed{W_t + \Lambda W_x = 0\,.} \quad \cdots (**)$$

→ Here, W : char. variables.

→ Since Λ is diagonal, $(**)$ is now a decoupled system of m indep. scalar eqns

$$\boxed{\frac{\partial w_k}{\partial t} + \lambda_k \frac{\partial w_k}{\partial x} = 0}, \quad k = 1, \cdots, m$$

$(***)$

where

$$W = \begin{Bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{Bmatrix}.$$

→ Likewise, $U = \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{Bmatrix}$ $\left(\text{e.g. } \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{Bmatrix} \right)$

→ Since (***) is a constant coeff. linear advection eqn individually, we can solve this using the char.;

$$w_k(x,t) = w_k(x - \lambda_k t, 0).$$

→ Once solving this for w_k, $k = 1, \cdots, m$, U can be recovered by

<u>projecting W back to U</u>

by multiplying W by R;

$$U = RW = [r_1 | r_2 | \cdots | r_m] \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

$$= \sum_{k=1}^{m} r_k w_k(x,t)$$

$$= \sum_{k=1}^{m} r_k w_k(x-\lambda_k t, 0) .$$

$\boxed{\text{Linearization of Nonlinear Systems}}$

→ So far, we've looked at the linear sys.

→ Now we look at a nonlinear sys. of conservation laws:

$$U_t + F(U)_x = 0 ,$$

$$U : \mathbb{R}\times\mathbb{R}^+ \longrightarrow \mathbb{R}^m , \quad F : \mathbb{R}^m \longrightarrow \mathbb{R}^m ,$$

$$(x,t) \rightsquigarrow \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$$

→ This can be written as

$$U_t + A(U)\,U_x = 0 , \quad \text{where}$$

$$A(U) = \frac{\partial F(U)}{\partial U} \; ; \; m\times m \text{ flux Jacobian matrix.}$$

→ Again, the system is (strictly) hyperbolic if $A(U)$ is diagonalizable with (distinct) real eigenvalues.

→ Now we want to linearize this system to numerically solve it.

→ A full linearization is possible using a local linearization about a constant state $\bar{U} = U_{avg}$,

and hence obtain a constant coeff. linear system, with A is frozen at U_{avg};

$$A(\bar{U}) = A(U_{avg}),$$

$$\Rightarrow \boxed{U_t + A(U_{avg})\, U_x = 0}$$

→ Here, the constant state

U_{avg} : an averaged state between U_L & U_R.

(Ex) $U_{avg} = \frac{1}{2}(U_L + U_R)$

(ex) Roe averages

Rmk → The "nonlinear" behavior of the linearized system would depend on the choice of U_{avg}.

→ ∞-many choices of writing U_{avg}.

→ The simplest choice is the arithmetic average of U_L & U_R.

The Euler Eqns.

→ We will look at the Euler Eqns in three different forms

⇒ (i) Conservative form (U)

(ii) Primitive form (V)

(iii) Characteristic form (W)

→ First let U : conservative variable

$$U = \begin{Bmatrix} \rho \\ \rho u \\ E \end{Bmatrix}, \quad E = \rho\left(\frac{u^2}{2} + e\right) : \text{total energy per unit volume}$$

→ EOS (caloric EOS) ;

$$e = e(\rho, p) = \frac{p}{\rho(\gamma - 1)}$$

→ Second, let V be the primitive var

$$V = \begin{Bmatrix} \rho \\ u \\ p \end{Bmatrix}.$$

→ Lastly, W = characteristic var that can be derived from either

(i) conservative var U ;

$$W = (R^c)^{-1} U = (L^c) U$$

$$= (l_1^c, l_2^c, l_3^c) \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix},$$ or

(ii) primitive variable V ;

$$W = (R^p)^{-1} V = (L^p) V$$

$$= (l_1^p, l_2^p, l_3^p) \begin{pmatrix} \rho \\ u \\ \phi \end{pmatrix}$$

[II] The Conservative variable form

$\rightarrow$ $U_t + F(U)_x = 0$, or

$$U_t + A(U)U_x = 0, \qquad A(U) = \frac{\partial F}{\partial U}$$

$\rightarrow$ $$F(U) = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{pmatrix},$$

EoS: $e = e(\rho, p) = \dfrac{p}{\rho(\gamma-1)}$, $\quad c_s = \sqrt{\dfrac{\gamma p}{\rho}}$.

$\rightarrow$ $$A(U) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial u_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{u^2}{2}(\gamma-3) & (\gamma-3)u & (\gamma-1) \\ \frac{u^3}{2}(\gamma-2) - \frac{c_s^2 u}{\gamma-1} & \frac{u^2(3-2\gamma)}{2} + \frac{c_s^2}{\gamma-1} & \gamma u \end{bmatrix}$$

Hint: use $p = \rho e(\gamma-1) = (\gamma-1)\left(E - \dfrac{(\rho u)^2}{2\rho}\right) = (\gamma-1)\left(u_3 - \dfrac{u_2^2}{2u_1}\right)$

$\rightarrow$ Diagonalizing A: $\quad A = R^c \Lambda L^c$, or

$$L^c A R^c = \Lambda, \qquad R = L^{-1}$$

$\rightarrow$ $W_t + \Lambda W_x = 0$.

$$\rightarrow \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$$

$$= (u - c_s,\ u,\ u + c_s),$$

$$R^c = \begin{array}{c} \begin{bmatrix} -\dfrac{\rho}{2c_s} & 1 & \dfrac{\rho}{2c_s} \\[2ex] -\dfrac{\rho}{2c_s}(u - c_s) & u & \dfrac{\rho}{2c_s}(u + c_s) \\[2ex] -\dfrac{\rho}{2c_s}\left(\dfrac{u^2}{2} + \dfrac{c_s^2}{\gamma - 1} - c_s u\right) & \dfrac{u^2}{2} & \dfrac{\rho}{2c_s}\left(\dfrac{u^2}{2} + \dfrac{c_s^2}{\gamma - 1} + c_s u\right) \end{bmatrix} \\ \begin{array}{ccc} \lambda_1 \qquad\qquad\qquad & \lambda_2 & \qquad\qquad\qquad \lambda_3 \end{array} \end{array}$$

$$L^c = \frac{\gamma - 1}{\rho c_s} \begin{array}{cc} \begin{bmatrix} -\dfrac{u^2}{2} - \dfrac{c_s u}{\gamma - 1} & u + \dfrac{c_s}{\gamma - 1} & -1 \\[2ex] \dfrac{\rho}{c_s}\left(-\dfrac{u^2}{2} + \dfrac{c_s^2}{\gamma - 1}\right) & \dfrac{\rho u}{c_s} & -\dfrac{\rho}{c_s} \\[2ex] \dfrac{u^2}{2} - \dfrac{c_s u}{\gamma - 1} & -u + \dfrac{c_s}{\gamma - 1} & 1 \end{bmatrix} & \begin{array}{c} \lambda_1 \\[4ex] \lambda_2 \\[4ex] \lambda_3 \end{array} \end{array}$$

Note:

$$f_1 = \rho u = u_2$$

$$f_2 = \rho u^2 + p = \frac{(\rho u)^2}{\rho} + \rho e(\gamma-1)$$

$$= \frac{u_2^2}{u_1} + (\gamma-1)\left(E - \frac{(\rho u)^2}{2\rho}\right)$$

$$= \frac{u_2^2}{u_1} + (\gamma-1)\left(u_3 - \frac{u_2^2}{2u_1}\right)$$

$$f_3 = u(E+p) = \frac{u_2}{u_1}\left(u_3 + (\gamma-1)\left(u_3 - \frac{u_2^2}{2u_1}\right)\right)$$

$\boxed{2}$ The primitive-variable form

$\rightarrow$ $V_t + A(V)\,V_x = 0$,

$$A(V) = \left[\frac{\partial f_i}{\partial V_j}\right]_{i,j} = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho c_s^2 & u \end{bmatrix}$$

$\rightarrow$ Decoupling $A = R^p \Lambda L^p$, or

$\Lambda = L^p A R^p$, where

$$R^p = \begin{array}{c} \begin{bmatrix} -\frac{\rho}{2c_s} & 1 & \frac{\rho}{2c_s} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{\rho c_s}{2} & 0 & \frac{\rho c_s}{2} \end{bmatrix} \\ \begin{array}{ccc} \quad\lambda_1 & \quad\lambda_2 & \quad\lambda_3 \end{array} \end{array}$$

$$L^p = \begin{bmatrix} 0 & 1 & -\frac{1}{\rho c_s} \\ 1 & 0 & -\frac{1}{c_s^2} \\ 0 & 1 & \frac{1}{\rho c_s} \end{bmatrix}$$

→ Similarly as before, we can convert the primitive-var. form into char-var. form:

$$W_t + \Lambda W_x = 0.$$

Rmk Relationship between $A(V)$ & $A(U)$.

→ Note $dU = Q\,dV$, where

$$Q = \frac{dU}{dV} = \begin{bmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ \frac{u^2}{2} & \rho u & \frac{1}{\gamma-1} \end{bmatrix},$$

→ Similarly, $dV = Q^{-1} dU$, where

$$Q^{-1} = \frac{dV}{dU} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 \\ \frac{1}{2}(\gamma-1)\,u^2, & -(\gamma-1)u, & \gamma-1 \end{bmatrix}$$

→ Then by chain-rule,

$U_t + A(U) U_x = 0$ can be written as

$$Q V_t + A(U) Q V_x = 0, \quad \text{or}$$

$$V_t + \underbrace{Q^{-1} A(U) Q}_{=A(V)} V_x = 0$$

→ $A(V) = Q^{-1} A(U) Q$,

they are similar matrices,

(i) they have the same eigenvalues.

[3] The characteristic-variable form

→ $W_t + \Lambda W_x = 0$,

→ Completely decoupled eqn ;

$$\frac{\partial w_k}{\partial t} + \lambda_k \frac{\partial w_k}{\partial x} = 0, \quad k = 1, \cdots, m.$$

→ Can be solved using the linear scalar advection.

Riemann Problems for linearized Euler Eqns

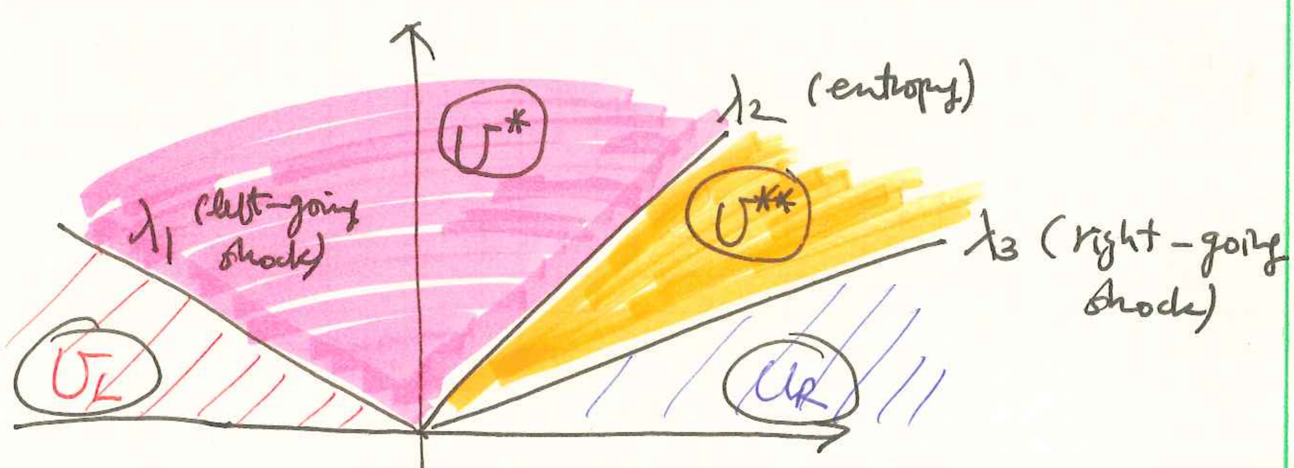
→ The (classical) RP :

$$\begin{cases} U_t + A U_x = 0 & \text{(PDE)} \\ U(x,0) = \begin{cases} U_L, & x<0 \\ U_R, & x>0 \end{cases} & \text{(IC)} \end{cases}$$

→ Let's assume $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ (strictly hyperbolic).

→ Further, let's assume $m=3$ for 1D Euler.

→ The structure of the soln of the RP in the x-t plane



→ m waves emanate from the origin, or in general, from each cell interface where the local RPs are considered.

→ Each k^{th} wave carries a jump discont. in U propagating with speed λ_k.

→ The goal is to find the soln in the Riemann fan region,

U^* & U^{**}, or

in more general, the region surrounded by U_L & U_R ; two initial states

→ In full 3D Euler;

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{pmatrix} \therefore m = 5,$$

In MHD : $U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ B_x \\ B_y \\ B_z \\ E \end{pmatrix}$, $m=7$

(it's 7, since $\nabla \cdot B = 0$)

→ Using right eigenvectors (linearly indep),

we write

$$U_L = \sum_{i=1}^{3} \alpha_i r_i ,$$

$$U_R = \sum_{i=1}^{3} \beta_i r_i ,$$

α_i, β_i : constant coeff.

→ Using $W = LU$, or $U = RW$,

$$U(x,t) = \sum_{k=1}^{m} r_k \omega_k (x,t)$$

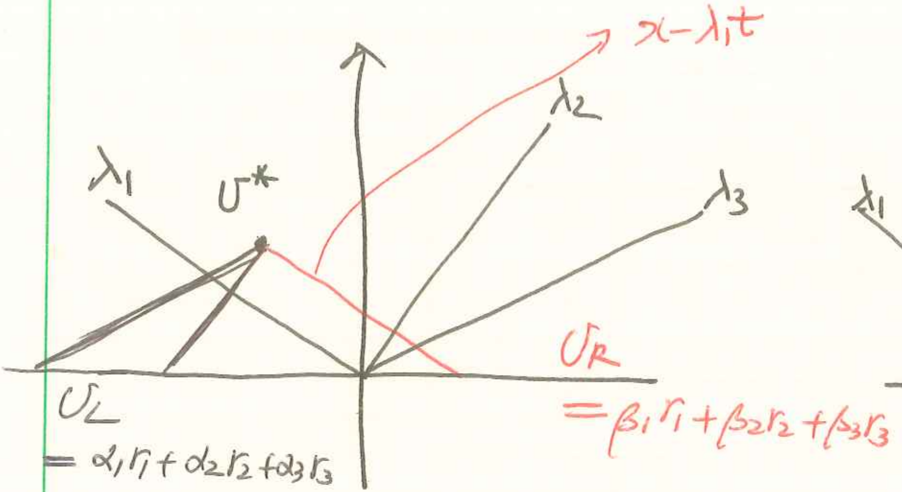
$$= \sum_{k=1}^{m} r_k \omega_k(x - \lambda_k t, 0) , \text{ where}$$

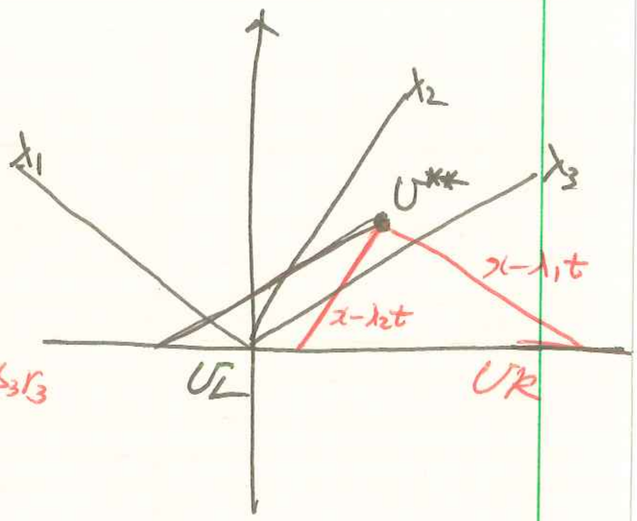
$$\omega_k(x-\lambda_k t, 0) = \begin{cases} \alpha_k , & x - \lambda_k t < 0 \\ \beta_k & x - \lambda_k t > 0 . \end{cases}$$

→ Letting I be the maximum value s.t

$x - \lambda_k t > 0$, $\forall k \le I$;

$$U(x,t) = \sum_{k=1}^{I} \beta_k r_k + \sum_{k=I+1}^{m} \alpha_k r_k$$





$U^* = \beta_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3$ $\qquad$ $U^{**} = \beta_1 r_1 + \beta_2 r_2 + \alpha_3 r_3$

→ Total jump ΔU across the whole wave structure :

$$\Delta U = U_R - U_L$$
$$= \sum_{k=1}^{3} (\beta_k - \alpha_k)\, r_k$$

→ The total jump is the sum of <u>individual jumps</u> across the kth waves, denoted by $\Delta U_k = (\beta_k - \alpha_k)\, r_k$,

where $\beta_k - \alpha_k$: the <u>strength of</u> the kth wave.

Characteristic Fields

→ The char. speed (or the eigenvalues) $\lambda_k = \lambda_k(U)$ defines a char. fields, the λ_k- field.

→ r_k- field: char. field defined by r_k.

Def. A λ_k- char. field is said to be linearly degenerate if

$$\nabla \lambda_k(U) \cdot r_k(U) = 0, \quad \forall U \in \mathbb{R}^m.$$

Def. A λ_k- field is genuinely nonlinear if

$$\nabla \lambda_k(U) \cdot r_k(U) \neq 0, \quad \forall U \in \mathbb{R}^m.$$

Rmks

$$\nabla \lambda_k(U) = \left(\frac{\partial \lambda_k}{\partial u_1}, \frac{\partial \lambda_k}{\partial u_2}, \frac{\partial \lambda_k}{\partial u_3} \right)^T,$$

for each k.

Ex The λ_2- field of the Euler is linearly degenerate

$\because$ $$\nabla \lambda_2(U) = \left(\frac{\partial \lambda_2}{\partial u_1}, \frac{\partial \lambda_2}{\partial u_2}, \frac{\partial \lambda_2}{\partial u_3}\right)^T$$

$$= \left(-\frac{u}{\rho}, \frac{1}{\rho}, 0\right),$$ since

$$\lambda_2 = u = \frac{\rho u}{\rho} = \frac{u_2}{u_1}$$

$$\rightarrow \nabla \lambda_2(U) = \left(-\frac{u_2}{u_1^2}, \frac{1}{u_1}, 0\right)$$

$$= \left(-\frac{\rho u}{\rho^2}, \frac{1}{\rho}, 0\right)$$

$$= \left(-\frac{u}{\rho}, \frac{1}{\rho}, 0\right). \checkmark$$

and $$r_2 = \left(1, u, \frac{u^2}{2}\right)^T$$

$$\Rightarrow \nabla \lambda_2 \cdot r_2 = 0 \quad \checkmark$$

(Ex) λ_1- & λ_3-fields are genuinely nonlinear.

Rmk. λ_k : linearly degenerate

$\Rightarrow$ λ_k is constant along each integral curve.

(ex) In a constant coeff. linear advection, $\lambda_k = \text{const.}$ everywhere, (i.e, $\lambda_k = a$)

(i) $\nabla \lambda_k(U) = 0$, $\forall U$.

(ex) In scalar nonlinear advection,

$$u_t + (f(u))_x = 0, \quad \text{with } m=1,$$

$\Rightarrow$ $\lambda_1(u) = f'(u) = \underset{\sim}{A}_{1\times 1}$

$\Rightarrow$ Since

$$\begin{array}{ccc} Ar & = & \lambda r \\ \downarrow & & \searrow \\ f'(u)r & & \lambda_1 r \\ \| & & \\ \lambda_1 r & & \end{array}$$

$\Rightarrow$ can take $r_1 = 1$

$\Rightarrow$ Genuinely nonlinear: $\underbrace{\nabla \lambda_1 \cdot r_1}_{= f''(u)} \neq 0$

$\Rightarrow$ convexity requirement.

Rmk. $\left\{\begin{array}{l} \lambda_2\text{-char. field : contact discont (always)} \\ \lambda_1 \text{ \& } \lambda_3 \text{ - fields : either rarefaction or shock} \end{array}\right.$

Elementary-wave solns of the RP

→ For nonlinear systems,

waves → discont (shock, contact)
↳ smooth (rarefactions)

→ Consider an elementary type of wave solns which consists of

U_L, U_R, that are connected by a single wave

→ RP : consists of only a single nontrivial wave, but not multiple of them
(ie, both shock & rarefaction, or both shock & contact discont)

→ This elementary wave will therefore be

{ shock or contact if discont, or
{ rarefaction if smooth.

$\boxed{1}$ Shock wave;

→ U_L & U_R are connected through a single jump discont in a genuinely nonlinear field k &

→ ① RH : $F(U_R) - F(U_L) = s(U_R - U_L)$

② $\lambda_k(U_L) > s > \lambda_k(U_R)$.

$\boxed{2}$ Contact wave;

→ U_L & U_R are connected through a single jump discont. of speed s in a linearly degenerate field k &

→ ① RH : $F(U_R) - F(U_L) = s(U_R - U_L)$

② Gen. Riemann. Invariants across the wave;

$$\frac{dw_1}{r_k \cdot e_1} = \frac{dw_2}{r_k \cdot e_2} = \frac{dw_3}{r_k \cdot e_3}, \quad \overset{m-1}{(\text{ODEs})}$$

◎ where w_1, w_2, w_3 : ~~char.~~ var,

$W = (w_1, w_2, w_3)^T$ either in conservative or primitive

◎ r_k : k^{th} right eig. vector (in cons, or prim)

◎ e_1, e_2, e_3 : unit vector $(e_3 = (0,0,1))$.

③ parallel char. condition ;

$$\lambda_k(U_L) = s = \lambda_k(U_R).$$

[3] Rarefaction Wave :

→ U_L, & U_R are connected through a smooth transition in a gen. nonlinear field k &

① Gen. Riemann Invariant

$$\frac{dw_1}{r_k \cdot e_1} = \frac{dw_2}{r_k \cdot e_2} = \frac{dw_3}{r_k \cdot e_3}$$

② the difference of characteristics

$$\lambda_k(U_L) < \lambda_k(U_R).$$

Ex) For r_2 - wave in the Euler eqn;

→ $\lambda_2 = u_2$

→ $r_2 = \begin{bmatrix} 1 \\ u \\ \frac{u^2}{2} \end{bmatrix}$ ← eigen of the conservative form

→ the Gen. Riemann Invariants across the r_2-wave;

$$W = U = (\rho, \rho u, E) = (w_1, w_2, w_3)$$

$$\frac{dw_1}{1} = \frac{dw_2}{u} = \frac{dw_3}{\frac{1}{2}u^2}$$

$$\rightarrow \frac{d\rho}{1} = \frac{d(\rho u)}{u} = \frac{d(E)}{\frac{1}{2}u^2} = c, \text{ const.}$$

$$\rightarrow \cancel{\begin{cases} d\rho = c \\ d(\rho u) = cu \\ dE = \frac{c}{2}u^2 \end{cases}} \Rightarrow \begin{cases} p = \text{constant} \\ u = \text{constant.} \end{cases}$$

(c) ① $u\,d\rho - d(\rho u) = 0$

Integrating; $\int u\,d\rho - \int d(\rho u) = c$

$\rightarrow \rho u - \rho u = c \quad , \quad c = 0$

$\rightarrow$ not useful.

② $\frac{1}{2}u^2 d\rho - dE = 0$

Integrating; $\int \frac{1}{2}u^2 d\rho - \int dE = c$

$\rightarrow \frac{\rho u^2}{2} - E = c \quad , \quad E = \rho(\frac{u^2}{2} + e)$

$\rightarrow \rho e = c \qquad\qquad e = \frac{p}{\rho(\gamma-1)}$

$\rightarrow \frac{p}{\gamma - 1} = c$

$\rightarrow \boxed{p = \text{const.}}$

③ $\frac{1}{2}u^2 d(\rho u) = u\,dE$

$\rightarrow \frac{1}{2}u\,d(\rho u) - dE = 0$

$\rightarrow$ Integrating; $\frac{1}{2}u(\rho u) - E = c$

$\rightsquigarrow$ same as ②.

(c') We actually get $(m-1)$ ODEs.

~~3) Since ρ = constant,~~

Also, 1) $d\rho = \text{const} \Rightarrow \int d\rho = \int \text{cons}\, d\rho$

$$\Rightarrow \rho = c \cdot \rho \;,\; c = 1$$

2) $d(\rho u) = u \cdot c$

→ Integrating w.r.t. ρu; $\int d(\rho u) = \int u \cdot c \, d\rho u$

$$\rho u = c u \cdot \rho u$$

$$\Rightarrow 1 = cu \,,\; u = \text{const}$$

3) $d(E) = \frac{c}{2} u^2$

→ Integrating both sides w.r.t. E

$$\int dE = \int \frac{c}{2} u^2 \, dE$$

$$\rightarrow \cancel{E} = \frac{c}{2} u^2 \cancel{E} \Rightarrow u^2 = \text{const}$$

$$\Rightarrow \underline{u = \text{const.}}$$

Approximate Riemann Solvers

→ We look at two solvers
(1) HLL
(2) Roe

→ They both are 1D-based Riemann solvers.

→ there are multidimensional Riemann solvers
(See work by Balsara, Dumbser et al.)

→ On the other hand, there are <u>exact</u> Riemann solvers for both <u>hydro</u> & <u>MHD</u>.
(often require iterations ⇒ expensive)

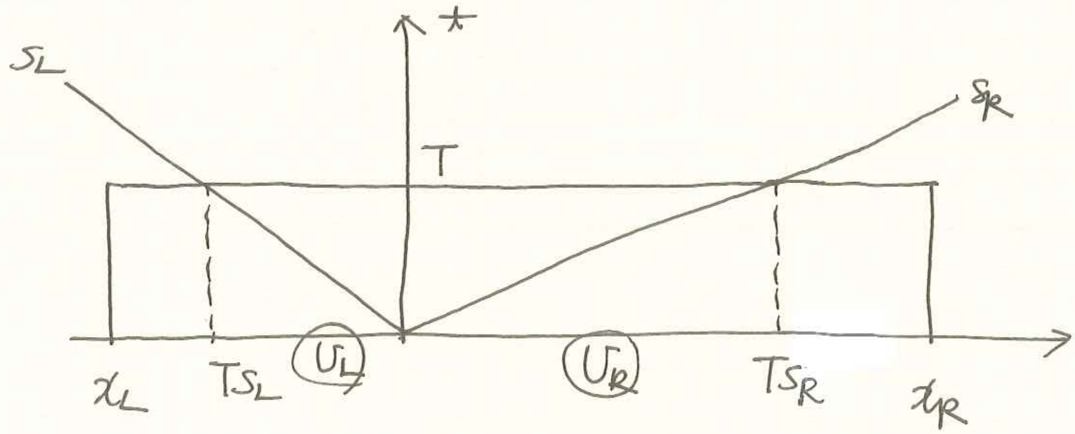
→ Approximate Riemann solvers are
(i) efficient (no iterations)
(ii) accurate, &
(iii) stable,

→ HLL & Roe : use local linearizations to replace the true nonlinear fluxes

[I] HLL (Harten – Lax – van Leer)

→ Consider a control volume

$[x_L, x_R] \times [0, T]$ on x-t plane



→ S_L & S_R : the fastest signal velocities arising from the soln of the RP.

→ the integral form of the conservation laws of $U_t + F(U)_x = 0$ becomes

$$\int_{x_L}^{x_R} U(x,T)\,dx - \int_{x_L}^{x_R} U(x,0)\,dx$$

$$= \int_0^T F(U(x_L,t))\,dt - \int_0^T F(U(x_R,t))\,dt$$

$$\rightarrow \boxed{\int_{x_L}^{x_R} U(x,T)\,dx = x_R U_R - x_L U_L + T(F_L - F_R)}$$ consistency condition

$F_{L;R} = F(U_{L;R}) = F(U(x_{L;R}, t))$

$\rightarrow$ $LHS = \int_{x_L}^{x_R} U(x,T)\,dx$

$$= \int_{x_L}^{TS_L} U(x,T)\,dx + \int_{TS_L}^{TS_R} U(x,T)\,dx + \int_{TS_R}^{x_R} U(x,T)\,dx$$

$$= (TS_L - x_L)\,U_L + \int_{TS_L}^{TS_R} U(x,T)\,dx + (x_R - TS_R)\,U_R$$

$\rightarrow$ $RHS = -x_L U_L + x_R U_R + T(F_L - F_R)$

(!) $$\int_{TS_L}^{TS_R} U(x,T)\,dx = T(S_R U_R - S_L U_L + F_L - F_R)$$

$\rightarrow$ $$\underbrace{\frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} U(x,T)\,dx} = \frac{S_R U_R - S_L U_L + F_L - F_R}{S_R - S_L}$$

$\rightsquigarrow$ The integral average of the exact soln of the RP between the slowest & fastest signals at time T

$\rightsquigarrow$ The RHS is constant with all known values, provided that we know S_L & S_R

→ We denote the integral average by

$$U^{hll} = \frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} U(x,T)\,dx ,$$

→ $$U^{hll} = \frac{S_R U_R - S_L U_L + F_L - F_R}{S_R - S_L}$$

→ Now consider the integral form of the conservation laws on $[x_L, 0] \times [0, T]$:

$$\int_{x_L}^{0} U(x,T)\,dx - \int_{x_L}^{0} U(x,0)\,dx = \int_0^T F(U_L)\,dt - \int_0^T F_{0L}\,dt$$

$$= T(F_L - F_{0L}),$$

F_{0L} = flux along the t-axis

$$\| $$

$$\int_{x_L}^{TS_L} (U(x,T) - U(x,0))\,dx$$

$$+ \int_{TS_L}^{0} (U(x,t) - U(x,0))\,dx$$

$$\| $$

$$\int_{x_L}^{TS_L} \underbrace{(U_L - U_L)}_{\to 0}\,dx + \int_{TS_L}^{0} U(x,T)\,dx - \int_{TS_L}^{0} \overbrace{U(x,0)}^{= U_L}\,dx$$

$$= \int_{TS_L}^{0} U(x,T)\,dx + TS_L U_L$$

$$\rightarrow \int_{TS_L}^{0} U(x,T)\,dx = -TS_L U_L + T(F_L - F_{0L})$$

$$\rightarrow F_{0L} = F_L - S_L U_L - \frac{1}{T}\int_{TS_L}^{0} U(x,T)\,dx,$$

Likewise, considering $[0, x_R] \times [0, T]$;

$$F_{0R} = F_R - S_R U_R + \frac{1}{T}\int_{0}^{TS_R} U(x,T)\,dx$$

$\rightarrow$ We should have $F_{0L} = F_{0R}$;

$$\therefore T\left(F_L - F_R - S_L U_L + S_R U_R\right) = \int_{TS_L}^{TS_R} U(x,T)\,dx$$

$\rightarrow$ Equivalent to the consistency condition.

$\rightarrow$ So far, all relations are <u>exact</u>,

$\rightarrow$ The HLL flux formulation;

$$U(x,t) = \begin{cases} U_L, & x/t \le S_L \\ U^{hll}, & S_L \le x/t \le S_R \\ U_R, & x/t \geq S_R \end{cases}$$

→ Now one can obtain S_L & S_R using

$$S_L = \min\{ u_L - c_{s,L} ,\ u_R - c_{s,R} \} = \min\{(\lambda_1)_L, (\lambda_1)_R\}$$

$$S_R = \max\{ u_L + c_{s,L} ,\ u_R + c_{s,R} \} = \max\{(\lambda_3)_L, (\lambda_3)_R\}$$

→ The final goal is to construct an approximate flux at each intercell bdry

$$F_{i+\frac{1}{2}} = \begin{cases} F_L & , 0 \leq S_L \\ F^{hll} & , S_L \leq 0 \leq S_R \\ F_R & , 0 \geq S_R \end{cases}$$

→ F^{hll} can be determined :

(i) from $$F_{0L} = F_L - S_L U_L - \frac{1}{T}\int_{TS_L}^{0} \underbrace{U(x,T)}_{\equiv U^{hll}} dx$$

$$= F_L - S_L U_L + \frac{1}{T} T S_L U^{hll}$$

$$= F_L + S_L (U^{hll} - U_L) \quad , \text{ or}$$

(ii) From $$F_{0R} = F_R - S_R U_R + \frac{1}{T}\int_0^{TS_R} \underbrace{U(x,T)}_{= U^{hll}} dx$$

$$= F_R + S_R (U^{hll} - U_R) .$$

→ If we let $F^{hll} = F_{0L} = F_{0R}$, then

$$F^{hll} = F_L + S_L\left(\frac{S_R U_R - S_L U_L + F_L - F_R}{S_R - S_L} - U_L\right)$$

$$= \frac{\cancel{(S_R - S_L)} F_L + S_L S_R U_R - \cancel{S_L S_L U_L} + \cancel{S_L F_L} - S_L F_R}{S_R - S_L}$$

$$\frac{- S_L (S_R - \cancel{S_L}) \cancel{U_L}}{S_R - S_L}$$

$$= \frac{S_R F_L - S_L F_R + S_L S_R (U_R - U_L)}{S_R - S_L} \checkmark$$

→ Note ; we did NOT do

$$F^{hll} = F(U^{hll})$$

→ This will be unstable because the lack of numerical diffusion.

→ Note ; HLL uses only S_L & S_R

⇒ bit diffusive in the Riemann fan soln.

⇒ HLLC improves the situation.