

3 Other variables and relations

3-1 Entropy is

Consider $dQ = de + pdv$, where

$$e = e(p, v) = \frac{pv}{\gamma - 1}$$

We will convert the RHS, $de + pdv$, to an exact DE by using an integrating factor $\frac{1}{T}$, that is,

Claim: $\frac{de + pdv}{T}$ is exact.

Note a DE

$$M(x, y)dx + N(x, y)dy = 0 \quad \text{is exact}$$

$$\text{iff } \exists \psi = \psi(x, y) \text{ s.t. } \left\{ \begin{array}{l} \psi_x = M \\ \psi_y = N \end{array} \right\} \quad \text{(Def)}$$

$$\text{iff } M_y = N_x \quad \text{(Theorem)}$$

$$\Rightarrow 0 = \psi_x dx + \psi_y dy = d(\psi)$$

$\Rightarrow \psi = c$ is a soln to the exact DE.

proof, To prove $\frac{de + pdv}{T}$ is exact,
we first see ;

$$(i) \quad de = \frac{\partial e}{\partial v} dv + \frac{\partial e}{\partial p} dp$$

$$= \frac{p}{\gamma-1} dv + \frac{v}{\gamma-1} dp$$

(ii) Hence,

$$de + pdv = \frac{p}{\gamma-1} dv + \frac{v}{\gamma-1} dp$$

$$\left(T = \frac{pv}{R} \right) \quad = \frac{1}{\gamma-1} (\gamma p dv + v dp)$$

$$v = \frac{1}{\rho} \quad (iii) \quad \frac{de + pdv}{T} = \frac{R\rho}{p} \frac{1}{\gamma-1} (\gamma p dv + v dp)$$

$$\boxed{ \rho d\left(\frac{1}{\rho}\right) = -\frac{d\rho}{\rho} }$$

$$(i) \quad d\left(\frac{1}{\rho}\right) = -\frac{d\rho}{\rho^2}$$

$$= \frac{R}{\gamma-1} (\gamma \rho dv + \frac{dp}{\rho})$$

$$= \frac{R}{\gamma-1} (\gamma \rho d\left(\frac{1}{\rho}\right) + \frac{dp}{\rho})$$

$$= \frac{R}{\gamma-1} \left(\frac{dp}{\rho} - \gamma \frac{d\rho}{\rho} \right)$$

$$= \frac{R}{\gamma-1} d \left[\ln\left(\frac{\rho}{\rho^\gamma}\right) \right] \quad \checkmark$$

$$d \ln\left(\frac{p}{\rho^\gamma}\right)$$

$$= d[\ln p - \gamma \ln(\rho)]$$

$$= \frac{dp}{p} - \gamma \frac{d\rho}{\rho}$$

\Rightarrow therefore, it is exact,

Now, since $\frac{de + pdv}{T}$ is exact,

$$\exists \text{ s.t. } ds = \frac{de + pdv}{T}$$



Def: We call s s.t.

the entropy.

$$ds = \frac{de + pdv}{T}$$

Equivalently, we write

$$Tds = de + pdv$$

Rank, $ds = \left(\frac{R}{\gamma-1}\right) d\left(\ln\left(\frac{P}{\rho^\gamma}\right)\right)$ $\left(\begin{array}{l} \text{specific heat} \\ \text{at constant} \\ \text{volume} \end{array}\right) = c_v, \text{ constant}$

$$= c_v d\left[\ln\left(\frac{P}{\rho^\gamma}\right)\right]$$

$$\Rightarrow S = c_v \ln\left(\frac{P}{\rho^\gamma}\right) + \text{const.}$$

\Rightarrow Solving for p :

$$\text{const} + \frac{S}{c_v} = \ln\left(\frac{P}{\rho^\gamma}\right)$$

$$\Rightarrow \frac{P}{\rho^\gamma} = k e^{S/c_v} \quad k: \text{const}$$

$$\Rightarrow \boxed{P = k e^{S/c_v} \cdot \rho^\gamma}$$

\Rightarrow using $p = k e^{s/cv} f^{\gamma}$ in the Euler Eqn, we can recast the Euler Eqn into (1D for example)

$$\begin{cases}
 \rho_t + (\rho u)_x = 0 \\
 (\rho u)_t + (\rho u^2 + p)_x = 0 \\
 \boxed{\rho_t + u \rho_x = 0} \leftarrow \text{non-conservative}
 \end{cases}$$

\Rightarrow We see here that the energy eqn is replaced with the entropy eqn

\Rightarrow EoS then can be s.t

$$p = p(f, s) \quad (\text{or } p(v, s))$$

$$\Rightarrow \left. \frac{\partial p}{\partial f} \right|_{s=\text{const}} = c_s^2, \quad c_s: \text{sound speed}$$

has a special meaning, which is

$$\begin{aligned} \Rightarrow \left. \frac{\partial p}{\partial \rho} \right|_{s=\text{const}} &= \gamma k e^{s/cv} \rho^{\gamma-1} \\ &= \gamma \left(k e^{s/cv} \cdot \rho^{\gamma} \right) / \rho \\ &= \frac{\gamma p}{\rho} \end{aligned}$$

$$\Rightarrow c_s = \sqrt{\frac{\gamma p}{\rho}}$$

|| def

$$\sqrt{\left. \frac{\partial p}{\partial \rho} \right|_{s=\text{const}}}$$

Rmk. From $S_t + \underbrace{u}_{\text{red circle}} S_x = 0!$

(i) In smooth flow, S remains constant on each particle path (i.e., along each streamline (i.e., tangent to the velocity vector, or along the trajectory of u in this case))

(ii) When a particle crosses a shock, then S may jump, but must increase!

\Rightarrow physical entropy condition

(iii) Roughly speaking, S measures the disorder in the system.

$$\text{i.e., } ds = \frac{de + pdv}{T} \geq 0$$

all the time.

(\Rightarrow 2nd Law of thermodynamics)

(iv) Examples of entropy-generating mechanism :

Ⓐ heat transfer, &

Ⓑ viscosity

within the internal structure of shock waves.

(v) Along a particle path in smooth flow, since S is constant,

$$p = \underbrace{k e^{\frac{s}{c_v}}}_{=\hat{k}} \rho^\gamma = \hat{k} \rho^\gamma \quad (\text{or } p \rho^\gamma = \hat{k})$$

\Rightarrow Isentropic flow

adiabatic & reversible
(no heat transfer) (no dissipation)

In general, we write

$$\boxed{pv^n = c}, \quad c: \text{const.}$$



polytropic process, n : polytropic index

Ex

⇒ Isentropic if $n = \gamma$ ($pv^\gamma = c$)

⇒ Isothermal if $n = 1$ (constant temperature)
 $c = pv = \underbrace{\left(\frac{\gamma}{\gamma-1}\right)}_{\substack{\# \text{ of molecules} \\ \text{per unit mass}}} RT$

⇒ Isochoric if $n = \infty$ (constant volume)
($v = c$)

⇒ Isobaric if $n = 0$ (constant pressure)
($p = c$)

For Isoentropic flows,

$$\Rightarrow p = \hat{k} \rho^\gamma$$

$$\Rightarrow E = \frac{p}{\gamma-1} + \frac{1}{2} \rho |\vec{v}|^2$$

$$= \frac{\hat{k} \rho^\gamma}{\gamma-1} + \frac{1}{2} \rho |\vec{v}|^2$$

~~$$= E(\rho, p, \vec{v})$$~~

$$= E(\rho, \rho \vec{v}) \leftarrow \text{no explicit dependency on } p$$

\Rightarrow E can be directly computed from the density and momenta eqns

\Rightarrow We can drop the energy eqn

$$\Rightarrow \begin{pmatrix} \rho \\ \rho u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + \hat{k} \rho^\gamma \end{pmatrix}_x = 0 \quad \text{in 1D.}$$

\Rightarrow The governing Eqn is simpler than the Euler eqn.

For Isothermal flow

$\Rightarrow \gamma = 1$, a useful case for illustrating the theory later, since the algebra is relatively simple, and yet the behavior is similar to the Euler eqns.

$\Rightarrow \boxed{T = \bar{T}}$, const. &

$$\boxed{p = \rho R T = \rho R \bar{T} = \rho \left(\frac{p}{\rho} \right) \stackrel{\gamma=1}{=} \rho \left(\frac{\gamma p}{\rho} \right)}$$
$$= \rho c_s^2, \quad c_s = \text{sound speed}$$

\Rightarrow To maintain a constant temperature on the system, there should be a constant heat flux through the boundary of the system

\Rightarrow Therefore, energy in the system is no longer conserved.

\Rightarrow But mass & momentum are still conserved in the system.

$$\Rightarrow \begin{pmatrix} \rho \\ \rho u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + \rho c_s^2 \end{pmatrix}_x = 0$$

\Rightarrow Isothermal eqn. ; simpler than
the Euler
eqns.

3-2 Specific heats, enthalpy.

→ In general, when an addition of heat dQ changes the temperature by dT ,

the ratio $c = \frac{dQ}{dT}$ is called

the heat capacity of the system

→ Consider $e = e(p, v)$, and

$$dQ = de + pdv.$$

→ Two cases when adding energy into the system:

Case ① keep $p = \text{const}$ and let v vary,
or

Case ② keep $v = \text{const}$ and let p vary.

→ Case 1

$$dQ = de + pdv \stackrel{p = \text{const.}}{=} de + d(pv) = d(e + pv)$$

We now define

$$h = e + pv : \text{specific enthalpy.}$$

$$\Rightarrow dQ = dh.$$

⇒ The heat capacity c_p at constant pressure now can be defined:

$$c_p = \frac{dQ}{dT} = \frac{dh}{dT}$$

⇒ Also since $Tds = dQ = dh$,

$$c_p = \frac{dh}{dT} = \frac{Tds}{dT}$$

⇒ Assuming $h = h(T, p)$, and likewise

$$s = s(T, p)$$

$$\begin{cases} dh = \frac{\partial h}{\partial T} dT + \frac{\partial h}{\partial p} dp \\ ds = \frac{\partial s}{\partial T} dT + \frac{\partial s}{\partial p} dp \end{cases}$$

$$\Rightarrow c_p = \left. \begin{cases} \frac{dh}{dT} = \left. \frac{\partial h}{\partial T} \right|_{p=\text{const}}, \text{ or} \\ \frac{Tds}{dT} = T \left. \frac{\partial s}{\partial T} \right|_{p=\text{const}} \end{cases} \right.$$

Case 2 $dQ = de + p dv \xrightarrow{0} = de$
 \parallel
 $T ds$

$$\Rightarrow C_v = \frac{dQ}{dT} = \frac{de}{dT} = \frac{T ds}{dT}$$

\Rightarrow Now assuming $e = e(T, v)$,
 $s = s(T, v)$

$$\begin{cases} de = \frac{\partial e}{\partial T} dT + \frac{\partial e}{\partial v} dv \xrightarrow{0} \\ ds = \frac{\partial s}{\partial T} dT + \frac{\partial s}{\partial v} dv \xrightarrow{0} \end{cases}$$

$$\Rightarrow C_v = \frac{de}{dT} = \left. \frac{\partial e}{\partial T} \right|_{v=\text{const}}$$

$$\frac{T ds}{dT} = T \left. \frac{\partial s}{\partial T} \right|_{v=\text{const}}$$

The heat capacity at constant volume

3-3) Ideal gases

$$\textcircled{1} P = \rho RT \Leftrightarrow p v = RT : \text{thermal Eos}$$

$$\left(v = v(T, p) = \frac{RT}{p} \right)$$

$$\textcircled{2} e = \frac{p}{\rho(\gamma-1)} : \text{caloric Eos}$$

$$\Rightarrow \textcircled{1} + \textcircled{2} : e = \frac{RT}{\gamma-1} = e(T) \text{ only,}$$

rather than $e = e(T, v)$, or
 $e = e(T, p)$.

$$\Rightarrow de = \frac{\partial e}{\partial T} dT + \cancel{\frac{\partial e}{\partial v} dv} \rightarrow$$

becomes an ordinary differential;

$$\Rightarrow \frac{\partial e}{\partial T} = \frac{de}{dT}$$

$$\Rightarrow \text{but } \frac{de}{dT} = \left. \frac{\partial e}{\partial T} \right|_{v=\text{const}} = c_v$$

$$\textcircled{i} \quad de = c_v dT.$$

Since $e = \frac{RT}{\gamma - 1}$, $de = \frac{R}{\gamma - 1} dT$

$$\textcircled{ii} \quad c_v = \frac{R}{\gamma - 1}$$

Also, $h = e + pv$

$$= \frac{RT}{\gamma - 1} + RT$$

$$= RT \left(\frac{\gamma}{\gamma - 1} \right)$$

$$\Rightarrow c_p = \frac{dh}{dT} = \frac{R\gamma}{\gamma - 1}.$$

$$\Rightarrow c_p = \frac{R\gamma}{\gamma - 1} = \gamma c_v$$

$$\Rightarrow \gamma = \frac{c_p}{c_v} \quad ; \text{ratio of specific heats.}$$

Also, one sees that

$$C_p - C_v = \frac{R\gamma}{\gamma-1} - \frac{R}{\gamma-1}$$
$$= \frac{R(\gamma-1)}{\gamma-1} = R.$$

$$\Rightarrow \boxed{C_p - C_v = R}$$

Remark Let α be degrees of freedom.

→ each degree of freedom contributes an average energy of $\frac{1}{2}kT$ per molecule, k : Boltzmann's constant

→ $\frac{\alpha}{2}kT$: total contribution per molecule with α degrees of freedom

→ $\frac{\alpha}{2}nkT = \underbrace{e}_{\text{internal energy}}$, n : # of molecules per unit mass

→ Note $R = nk$ by def.

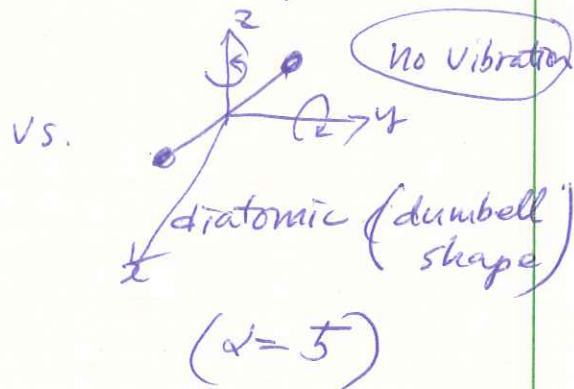
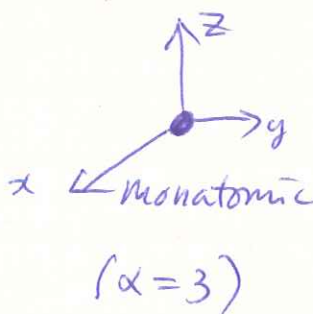
→ Comparing this with $e = C_v T$;

$$C_v = \frac{\alpha R}{2}$$

$$\begin{aligned} \rightarrow \text{Also, } C_p &= R + C_v \\ &= \left(1 + \frac{\alpha}{2}\right) R \end{aligned}$$

$$\rightarrow \boxed{\gamma = \frac{C_p}{C_v} = \frac{\alpha + 2}{2}}$$

(Ex) For monatomic gas, (mono + atomic) = single atom
the only degrees of freedom = three
translational degrees (x, y, & z)



$$\Rightarrow \gamma = \frac{3+2}{3} = \frac{5}{3} \approx 1.6666 \dots 7.$$

For diatomic,

$$\gamma = \frac{7}{5} = 1.4, \text{ (air)} \rightarrow \underline{N_2 + O_2}$$

3 translational

+ 2 rotational

(3 rotational axis minus one along the molecular axis)