Chapter 5.

Some Nonlinear Systems

→ We will have some examples in Nonlinear systems in this chapter.

→ Examples

① Euler equations

ⓐ Ideal gas — EoS.

ⓑ Entropy

② Isentropic flow

③ Isothermal flow

→ these examples will be used later to illustrate the nonlinear theory.

$\boxed{0}$ Units of measure

→ Will use the International System of Units (or Système International d'Unites, SI)

→ Traditionally, we require three basic units

① mass (M) – kg
② length (L) – meter
③ time (T) – second

$\Big\}$ "MKS" (meter kilogram second)

→ One also can vary the base of units to get cgs (centimeter-gram-second)

→ We treat MKS to be SI

→ Basic quantities

Quantity	Dimension
mass (M)	M
Area	L^2
Volume (V)	L^3
Time	T

→ Also, we see that

① force = mass x acceleration ($F = ma$)

② energy = force x distance

(or work "W" = force x distance moved in the direction of the force)

③ pressure = force / area

④ power = energy / time

⇒

Quantity	SI Unit	Dimension
$\rho = \frac{M}{V}$	kg/m^3	M/L^3
speed	m/s	L/T
acceleration	m/s^2	L/T^2
force	N (Newton) $kg \cdot m/s^2$	ML/T^2
pressure	N/m^2	M/LT^2
temperature	K (kelvin)	
energy (same for work)	Joule	ML^2/T^2
power	Watt	ML^2/T^3

[I] The Euler Equations

→ In conservation form;

$$\boxed{U_t + \nabla \cdot F(U) = 0}, \text{ where}$$

① $U = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}$ & $F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{pmatrix}$ for 1D

② $U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{pmatrix}$ & $F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ u(E+p) \end{pmatrix}$

$$G(U) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v w \\ v(E+p) \end{pmatrix}$$

$H(U) = \begin{pmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ w(E+p) \end{pmatrix}$ for 3D.

→ Note here that, in 1D,

if ∃ a quantity s that is advected with the flow

⇒ ∃ a contribution to the flux for s which is of the form su

→ thus, for example, the momentum equations in

x-dir has contributions of the form

(i) $\rho u * u$ in x-flux
(ii) $\rho u * v$ in y-flux
(iii) $\rho u * w$ in z-flux.

y-dir has contributions of the form

(i) $\rho v * u$ in x-flux
(ii) $\rho v * v$ in y-flux
(iii) $\rho v * w$ in z-flux

z-dir has contributions of the form

(i) $\rho w * u$ in x-flux
(ii) $\rho w * v$ in y-flux
(iii) $\rho w * w$ in z-flux

→ Additionally, besides advection,

⇒ ∃ forces on the fluid causing the fluid to accelerate

⇒ These forces are due to Newton's 2nd law $F = ma$

⇒ This Newton's Law principle needs to be added in momentum in each x, y, z-dir.

⇒ Without any external forces (e.g, gravitational forces, electromagnetic forces, etc);
the only force acting on the fluid is due to variations in the fluid itself

⇒ This force is proportional to $\boxed{\nabla p}$ (pressure gradient)

⇒ Note that ∇p is a vector quantity, which makes a perfect sense since force is also a vector quantity:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a},$$

$\vec{p}$: momentum of the system
$\vec{v} = (u, v, w)$: velocity fields; $\vec{a}$: acceleration

⇒ the contributions of ∇p therefore need to be included in each normal direction:

(i) $(\rho u^2)_x + p_x \quad \left(= (\rho u^2 + p)_x\right)$

(ii) $(\rho v^2)_x + p_y \quad \left(= (\rho v^2 + p)_y\right)$

(iii) $(\rho w^2)_x + p_z \quad \left(= (\rho w^2 + p)_z\right)$

→ Lastly, for the energy eqn:
there are contributions to the fluxes of the form

(i) $(E+p)u \longrightarrow [(E+p)u]_x$

(ii) $(E+p)v \longrightarrow [(E+p)v]_y$

(iii) $(E+p)w \longrightarrow [(E+p)w]_z$,

where the total energy E:

$$E = \frac{1}{2}\rho|\vec{v}|^2 + \rho e, \quad \rightsquigarrow \frac{M}{L^3}\left(\frac{L}{T}\right)^2 + \frac{M}{L^3}\cdot\frac{L^2}{T^2}$$

$$|\vec{v}|^2 = u^2 + v^2 + w^2,$$

→ kinetic energy

→ internal energy.

"specific" force × distance

$$M\frac{L}{T^2}\cdot L\cdot\frac{1}{M} = \frac{L^2}{T^2}$$

⇒ Note here that, without any external forces,

"<u>work</u>" is done only by the <u>pressure forces</u> which is proportional to $(pu)_x$, $(pv)_y$, & $(pw)_z$, in each x, y, z - direction.

⇒ I.e, in x- direction;

$$\underbrace{(Eu)_x}_{\substack{\text{work done}\\ \text{due to}\\ \text{advection in}\\ x\text{-dir}}} + \underbrace{(pu)_x}_{\substack{\text{work done due}\\ \text{to pressure}\\ \text{forces in}\\ x\text{-dir.}}}$$

<u>Rmk</u> <u>Unit check</u>

① $\rho u^2 + p$ (momentum flux in x-dir)

$\rho u^2 \to \frac{M}{L^3}\cdot\left(\frac{L}{T}\right)^2 = \frac{M}{LT^2}$ ✓ $\qquad p \to \frac{ML}{T^2}\cdot\frac{1}{L^2} = \frac{M}{LT^2}$ ✓

total energy

$\frac{1}{2}\rho|\vec{v}|^2 + \rho e$

$\frac{1}{2}\rho|\vec{v}|^2 \to \frac{M}{L^3}\left(\frac{L}{T}\right)^2 = \frac{M}{LT^2}$ $\qquad \rho e \to \frac{M}{L^3}\cdot\frac{L^2}{T^2} = \frac{M}{LT^2}$

② $u(E+p)$ (energy flux in x-dir)

$uE \to \frac{L}{T}\,\frac{M}{LT^2} = \frac{M}{T^3}$ ✓ $\qquad up \to \frac{L}{T}\cdot\frac{M}{LT^2} = \frac{M}{T^3}$ ✓

⇒ Note that the two energies are both multiplied by the density "ρ", therefore, "total" energies.

⇒ However, if you consider them without ρ, for example, e only;

e = the internal energy per unit mass
= the specific internal energy

per unit mass

⇒ e includes;

(i) microscopic kinetic energy due to microscopic motion of system's particles (e.g, translations, rotations, vibrations)

(ii) microscopic forces;
(e.g, chemical bonds between particles)

⇒ In the Euler eqn, we assume the gas is in chemical and thermodynamic equilibrium

⇒ $e = e(p, \rho) \overset{\text{OR}}{=} e(p, v)$, where $v = \frac{1}{\rho}$.

[specific volume, NOT velocity!!