Chapter 4

Examples of scalar conservation laws:

→ f is convex (or concave)

ex) Burgers Eqn: $u_t + \left(\frac{u^2}{2}\right)_x = 0$

ex) Traffic flow: $\rho_t + (\rho u)_x = 0$

(ρ: density of cars)

⇒ RP is always either a shock or a rarefaction

→ f is non-convex

ex) Two phase flow using Buckley-Leverett eqn

$$u_t + \left(\frac{u^2}{u^2 + a(1-u)^2}\right)_x = 0$$

⇒ RP might involve both a shock & a rarefaction

⇒ entropy condition needs to be different to take both.

Remark: Given u_L & u_R, the entropy conditions are different for
$$\begin{cases} f: \text{convex (or concave)} \\ f: \text{non-convex.} \end{cases}$$

	f: convex	f: non-convex (Oleinik)
Entropy condition ($u(x,t)$ is the entropy soln if all discont. have ⇒)	$f'(u_L) > s > f'(u_R)$	$\frac{f(u)-f(u_L)}{u-u_L} \geq s \geq \frac{f(u)-f(u_R)}{u-u_R}$ $\forall u$, between u_L & u_R

⇒ If f is convex, the both conditions are equivalent.

$\boxed{\text{Case 1}}$

$u_R < u_{max} < u_L$ $\Rightarrow$ compound wave traveling to the right

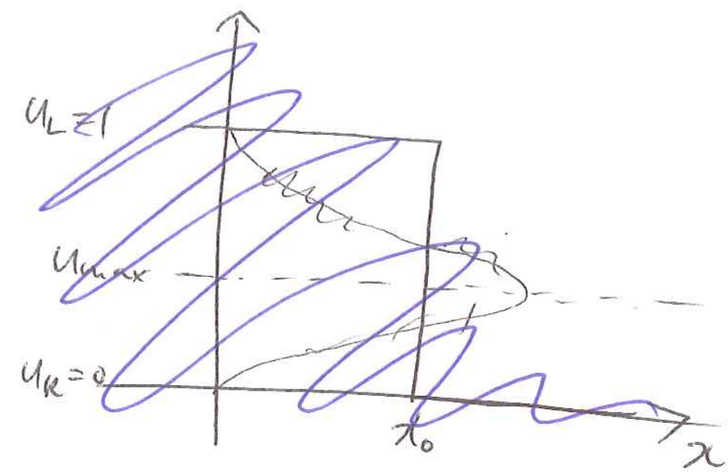
$\rightarrow$ $\lambda(u) = f'(u) = \dfrac{2au(1-u)}{[u^2+a(1-u)^2]^2}$ does not vary monotonically, between u_L & u_R.

A compound wave appears.

$\rightarrow$ We determine method of chars & equal area rule. to enforce uniqueness.

$\rightarrow$ ~~where we are interested in finding~~

~~We first note that~~

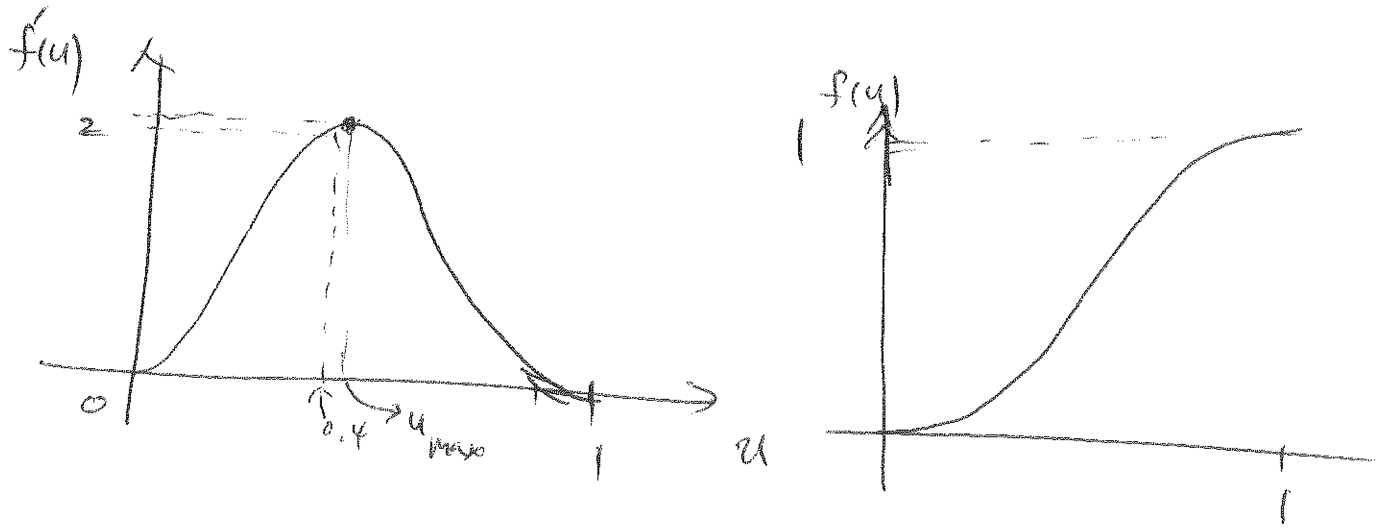


$\rightarrow$ Since char. velocities are $\lambda(u)=f'(u)$, the chars are given in the form of a func of u & t;

$$x = x(u,t) = x_0 + \lambda(u)t, \qquad x_0 = x(u,0).$$

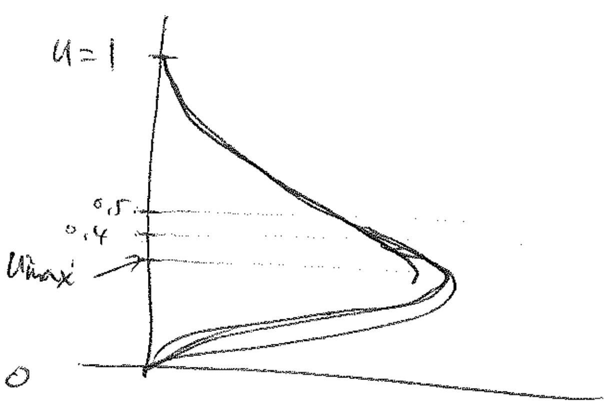
$$\Rightarrow x = x_0 + f'(u)\, t \quad ; \text{ chars.}$$
$$= x_0 + \lambda(u)\, t$$

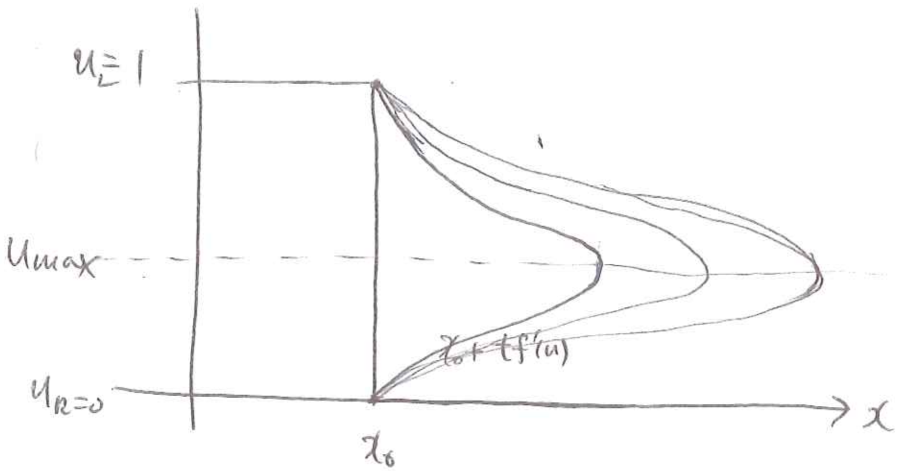
→ To see how it works, let's first look at the shape of $f(u)$, say for $a = \frac{1}{2}$;



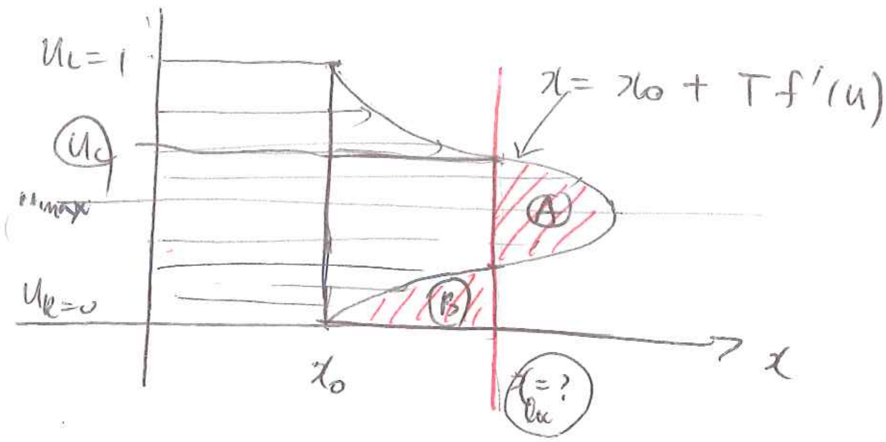
→ At any ~~given~~ time t, the flow develops according to the resulting weak soln that moves along the chars that looks like

$$x(u,t) = x_0 + \underline{\underline{f'(u)\, t}}$$





Equal area rule; At time T;



Let u_c be the value that ~~makes the~~ satisfies the equal area rule & for A & B;

$$\int_{u_L}^{u_R} x_0 + Tf'(u)\,du = \int_{u_L}^{u_c} x_0 + Tf'(u)\,du$$

$$+ \int_{u_c}^{u_R} (x_0 + x_{loc})\,du$$

$$\Rightarrow \int_{u_L}^{u_R} Tf'(u)\,du = \int_{u_L}^{u_c} Tf'(u)\,du + \int_{u_c}^{u_R} x_{loc}\,du$$

$$= \int_{u_L}^{u_c} Tf'(u)\,du + (u_R - u_c)\,x_{loc}$$

$\Rightarrow$ Since T is constant,

(i) $$T \int_{u_L}^{u_R} f'(u)\,du = T\,[\,f(u_R) - f(u_L)\,]$$

(ii) $$T \int_{u_L}^{u_c} f'(u)\,du = T\,[\,f(u_c) - f(u_L)\,]$$

$$\Rightarrow \quad x_{loc} = \frac{T}{u_R - u_c}\,[\,f(u_R) - f(u_c)\,]$$

$\#$

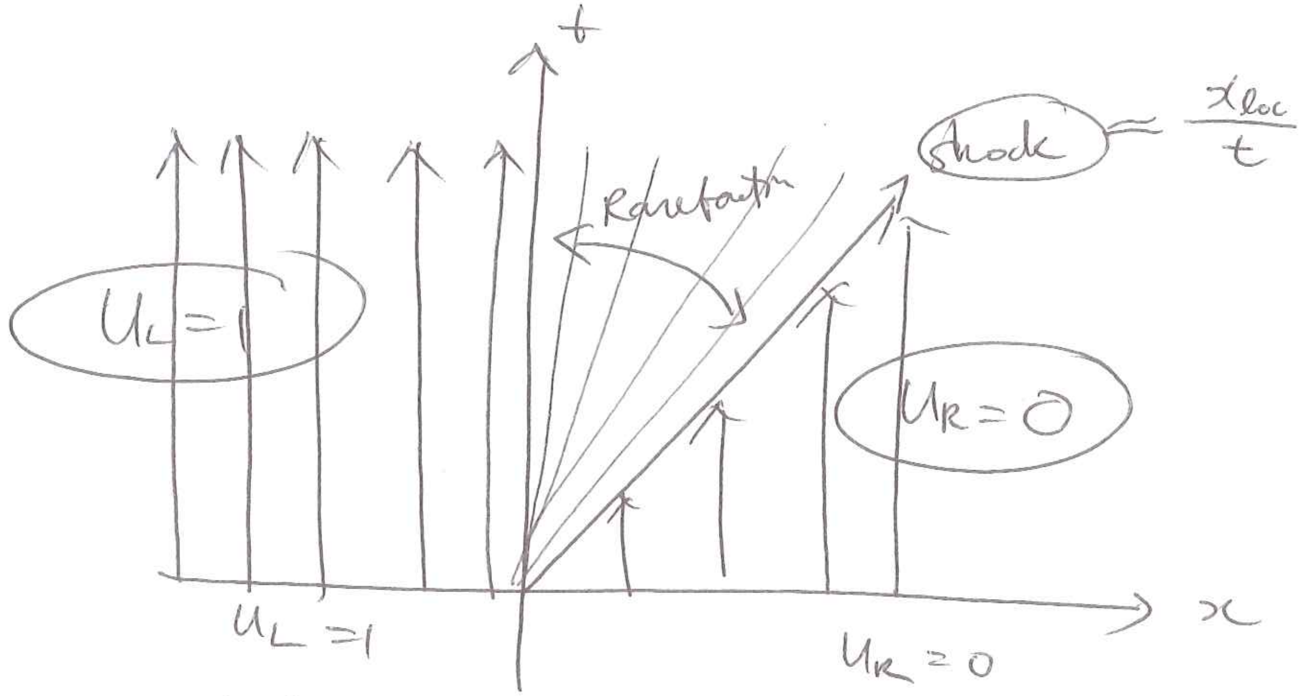
$\Rightarrow$ ~~[illegible]~~

$$\frac{f(u_R) - f(u_c)}{u_R - u_c} = \frac{x_{loc}}{T} \rightsquigarrow \text{shock speed.}$$

$\searrow$ RH condition

☺ RH condition is always satisfied.

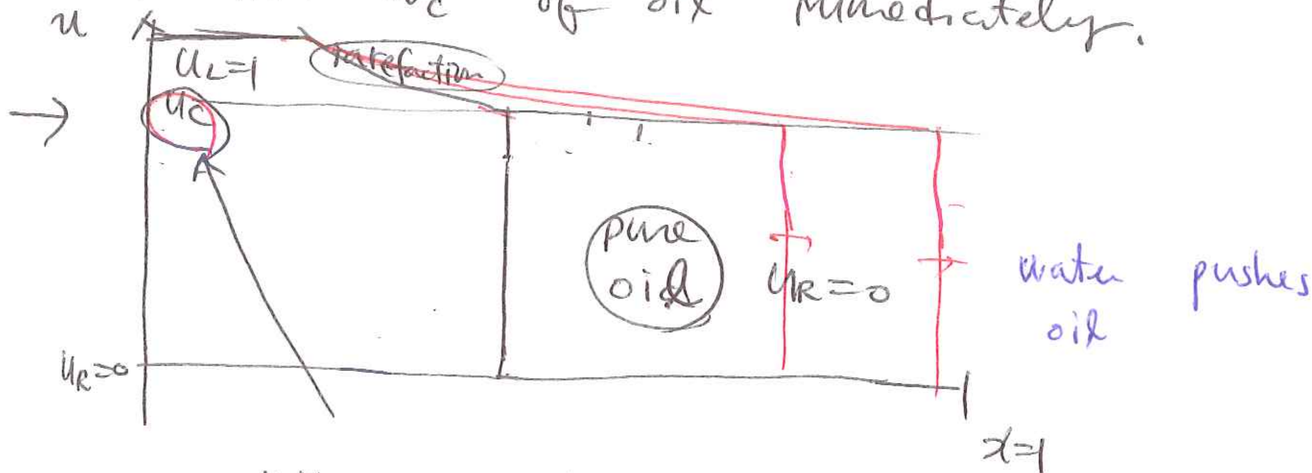
In $x-t$ plane;



$u_L = 1$ $\Rightarrow f'(u_L) = 0$

$u_R = 0$ $f'(u_R) = 0$

→ As water moves in, it displaces certain fraction u_c of oil immediately.



mixture of oil & water

(i.e, $0 < u_c < 1$) → constant in time (shown later why)

0 = pure oil, 1 = pure water

→ One obtains pure oil until the shock arrives,

followed by a ~~mix~~ mixture of oil & water
with diminishing returns as $t \to \infty$,

→ Note that the RP soln involves both

$\left\{\begin{array}{l}\text{a shock } \& \\ \text{a rarefaction}\end{array}\right.$

waves

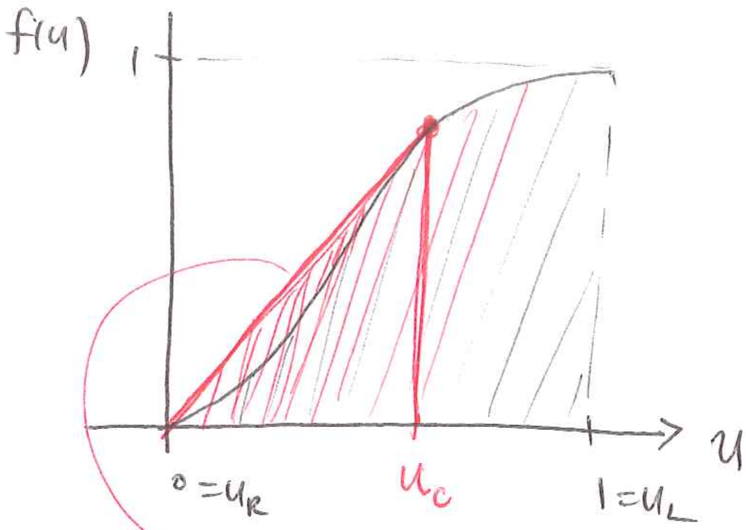
→ It turns out that the soln to the RP can be determined from the graph of f

→ If $u_R < u_L$: we use <u>convex hull</u> of

$$\{ (x,y) \mid u_R \le x \le u_L ,\ y \le f(x) \}.$$

= the smallest convex set containing the original set.

Convex Hull construction from $f(u)$



A straigh line segment from

$(u_R, f(u_R)) = (0,0)$ to $(u_c, f(u_c))$ &

$y = f(u)$ up to $(u_L, f(u_L))$.

$\Rightarrow$ ① The straight line represents a shock jumping from $u_L = 0$ to $u = u_c$.

② the line following $f(u)$; rarefaction wave.

$\Rightarrow$ Also note that the slope of the line segment is

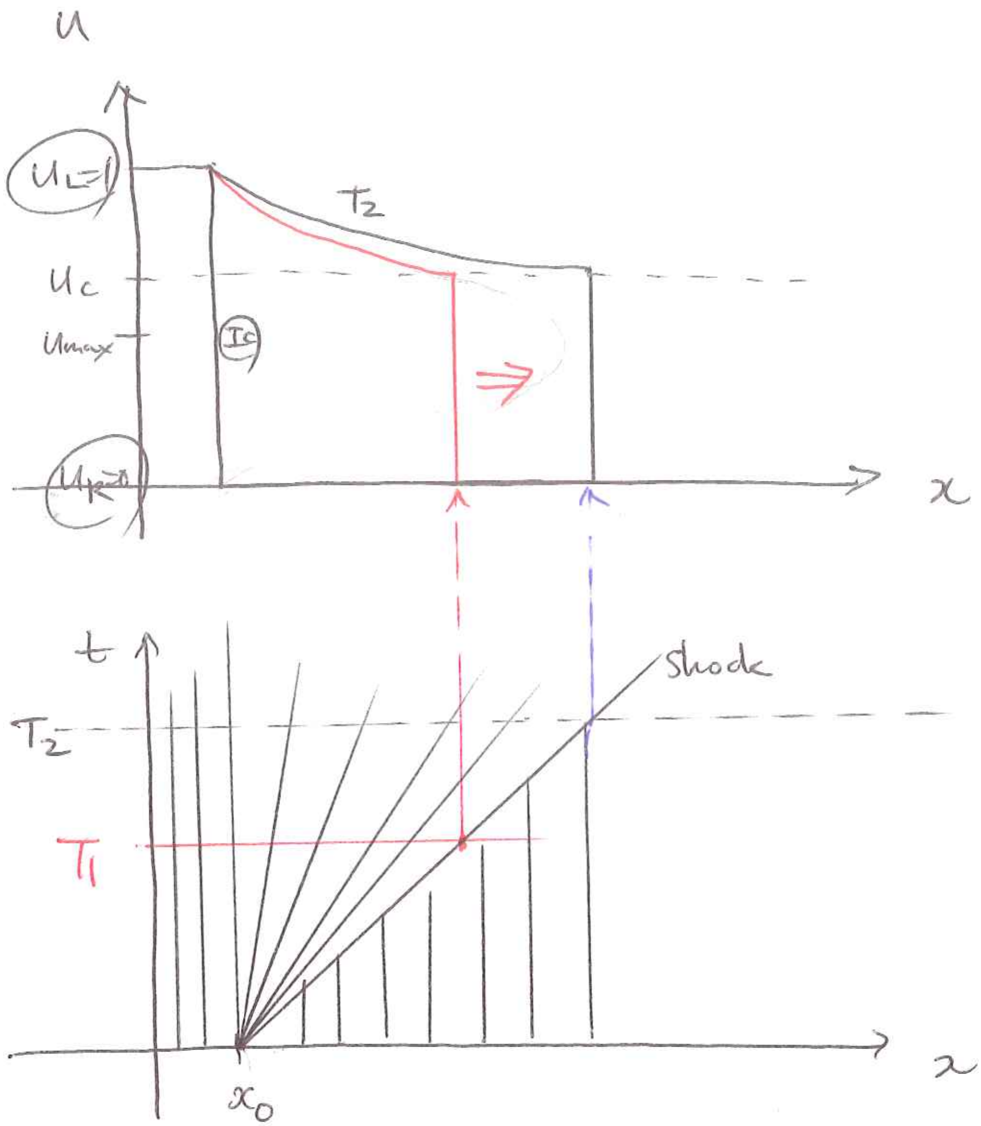
$$\frac{f(u_c) - f(u_R)}{u_c - u_R} \quad \left(= \frac{f(u_c)}{u_c} \right)$$

which is exactly the shock speed we've found.

$\Rightarrow$ This line is tangent to the curve $f(u)$ at u_c, (i.e, $f'(u_c)$), where

u_c is unique so that the convex hull is constructed for given $f(u)$,

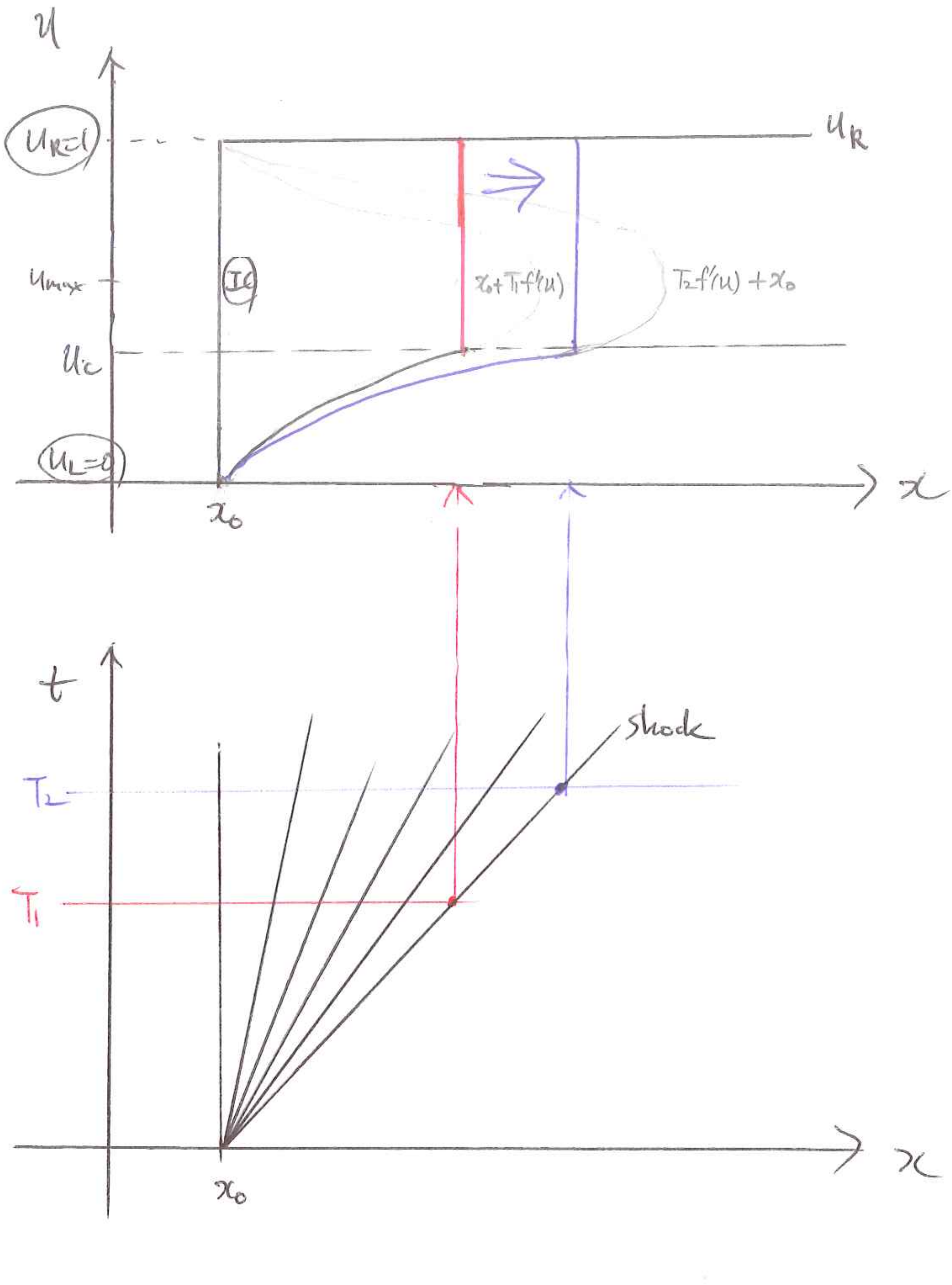
means that the shock moves at the same speed. as represented as the characteristics at the edge of the rarefaction fan.



$\boxed{\text{Case 2}}$ $u_L < u_{max} < u_R$

$\Rightarrow$ compound wave traveling to the right which can be calculated in the same way as in $\boxed{\text{case 1}}$

$\Rightarrow$



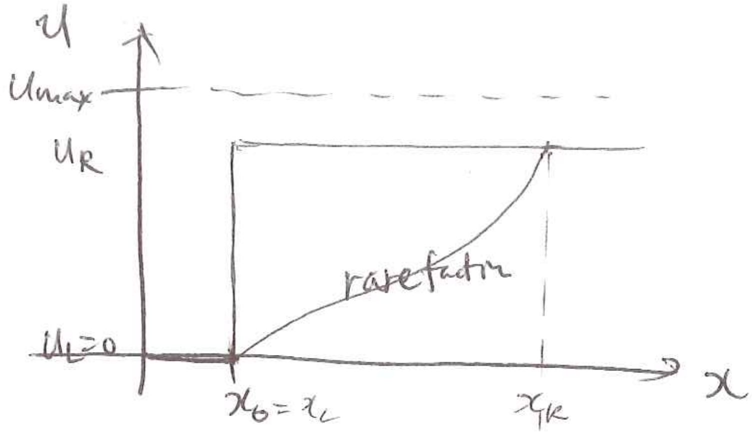
There are four more configurations that are simpler than the previous two cases.

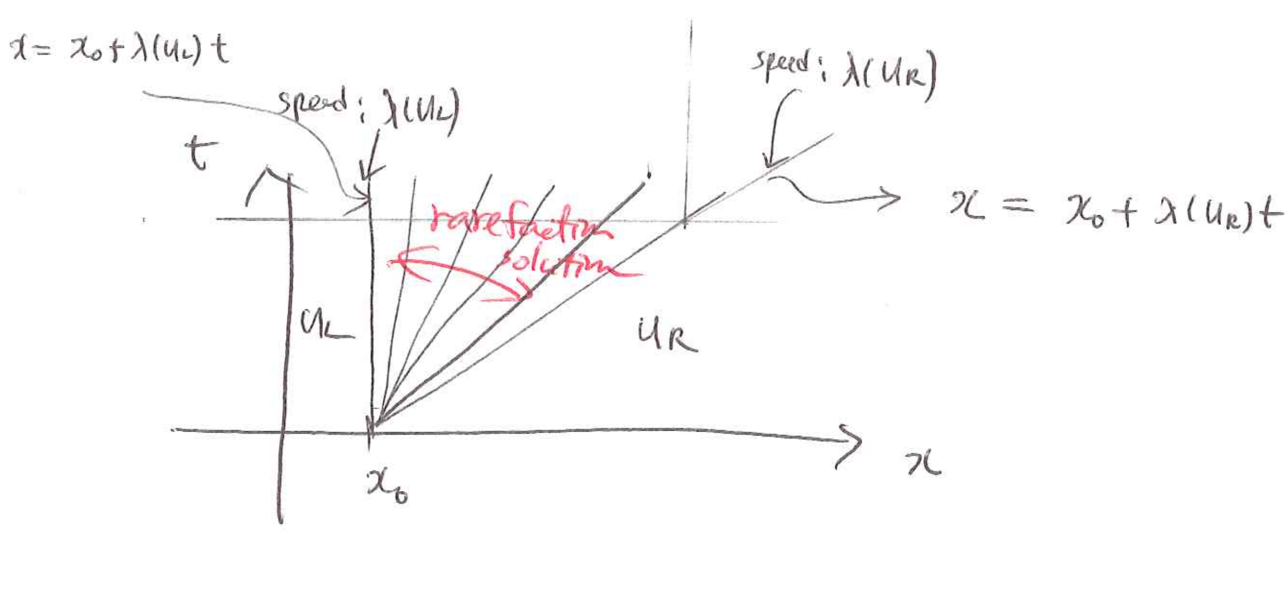
$\boxed{\text{Case 3}}$

$u_L < u_R \le u_{max}$

$\Rightarrow$ $f'(u_L) < f'(u_R)$ and $f'(u)$ varies monotonically between u_L & u_R.

$\Rightarrow$ Rarefaction solution appears which travels to the right.





→ The rarefaction soln satisfies

$$\frac{x-x_0}{t} = \lambda(u) = f'(u)$$

which connects between

$$\begin{cases} x_R = x_0 + \lambda(u_R)t & \text{and} \\ x_L = x_0 + \lambda(u_L)t & \text{for some } t. \end{cases}$$

→ Assuming we can find the inverse ftn of λ, the solution is

$$u(x,t) = \begin{cases} u_L & , \quad x \le x_L \\ \lambda^{-1}\left(\frac{x-x_0}{t}\right) & , \quad x_L < x \le x_R \\ u_R & , \quad x \ge x_R \end{cases}$$

$\boxed{\text{Case 4}}$ $u_R < u_L \le u_{max}$

$\Rightarrow \lambda(u_L) < \lambda(u_R)$

$$\begin{array}{ccc} \lambda(u_L) & < & \lambda(u_R) \\ \| & & \| \\ f'(u_L) & & f'(u_R) \end{array}$$

and f' (or λ) varies monotonically between u_L & u_R

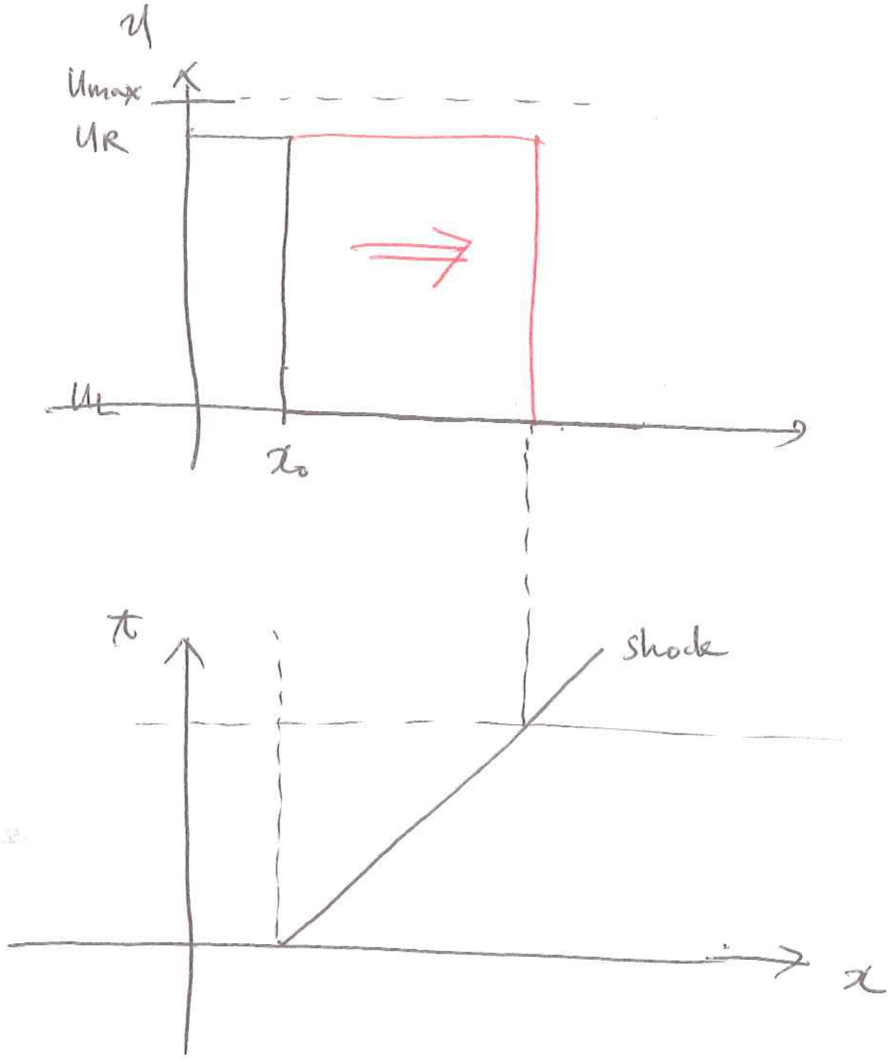
$\Rightarrow$ Shock solution is the weak soln.

$\Rightarrow$ shock travels with the speed to the right

$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R} \quad \text{by RH condition}$$

and the solution is given as

$$u(x,t) = \begin{cases} u_L, & x < x_0 + st \\ u_R, & x > x_0 + st \end{cases}$$



u
umax
uR
uL
x0
t
shock
x

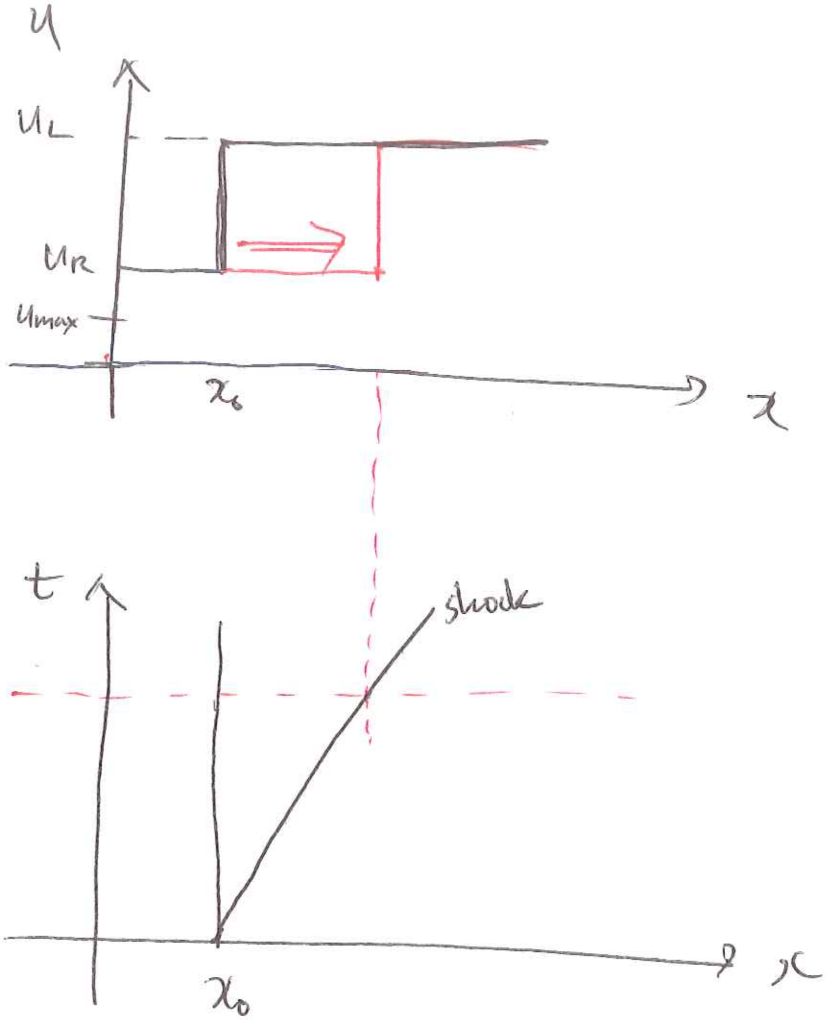
$\boxed{\text{case 5}}$

$$u_{max} \le u_L < u_R$$

$$\Rightarrow \quad \lambda(u_L) > \lambda(u_R)$$

$\Rightarrow$ <u>shock soln</u> as λ varies monotonically between u_L & u_R

$\Rightarrow$ Shock travels to the right with the same shock speed as in $\boxed{\text{case 4}}$.



$\boxed{\text{Case 6}}$ $\quad u_{max} \leq u_R < u_L$

$\Rightarrow \lambda(u_L) < \lambda(u_R)$, λ : varies monotonically between u_L & u_R

$\Rightarrow$ <u>rarefaction soln</u> is the weak soln.

between
$$\begin{cases} x_L = x_0 + \lambda(u_L)t \\ x_R = x_0 + \lambda(u_R)t \end{cases}$$

$\Rightarrow$ same as $\boxed{\text{case 3}}$.