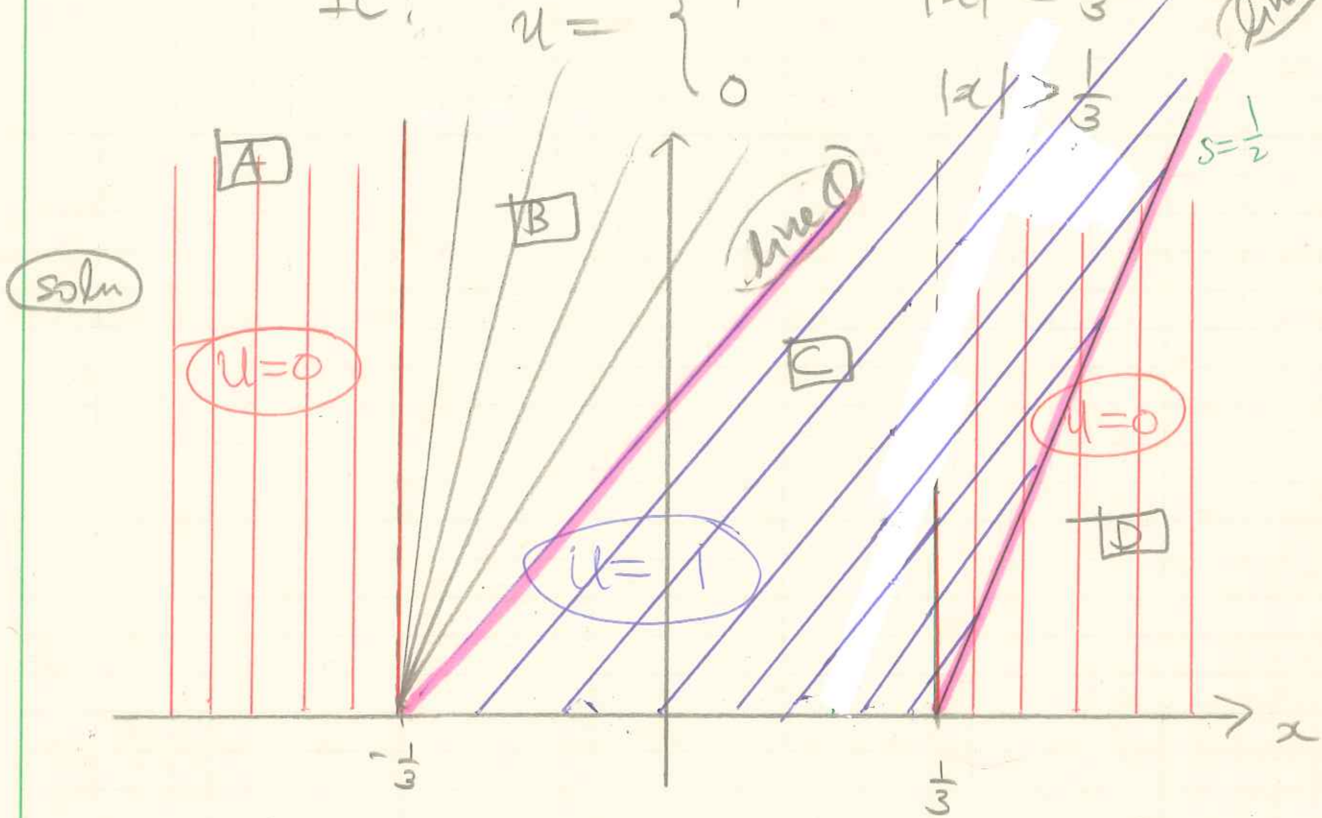
Ex Solve Burgers' Eqn on ∞ domain for $t \le \frac{4}{3}$ with

IC: $u = \begin{cases} 1 & |x| < \frac{1}{3} \\ 0 & |x| > \frac{1}{3} \end{cases}$



There are <u>two</u> jump discontinuities at

$x = -\frac{1}{3}$ & $x = \frac{1}{3}$. $\Rightarrow$ Three regions of solution.

[I] First consider @ $\boxed{x = -\frac{1}{3}}$

It is convenient to introduce $X = x + \frac{1}{3}$, so that the jump at $x = -\frac{1}{3}$ corresponds to the jump at $X = 0$.

$$\Rightarrow x = X - \frac{1}{3}.$$

The soln to the RP at $x = -\frac{1}{3}$:
(or, the soln to the RP at $x=0$)

IC $$\underset{\underset{u(x+\frac{1}{3},0)}{\parallel}}{u(x,0)} = \begin{cases} 0, & x<0 \quad (=u_L) \\ 1, & x>0 \quad (u_R) \end{cases} \qquad \left(f = \frac{u^2}{2}\right)$$

$$\Rightarrow s = \frac{[f]}{[u]} = \frac{\frac{1}{2} - 0}{1-0} = \frac{1}{2}$$

$\Rightarrow$ Since $u_L < u_R$, this will be a rarefaction soln. s.t. the diverging char. condition hold :

$$\begin{array}{ccc} \lambda(u_L) & < & \lambda(u_R) \\ \parallel & & \parallel \\ f'(u_L) & & f'(u_R) \\ \parallel & & \parallel \\ u_L & & u_R \\ \parallel & & \parallel \\ 0 & & 1 \end{array}$$

$$\Rightarrow \underset{t>0}{u(x,t)} = \begin{cases} 0, & \frac{x}{t} < 0 \\ \frac{x}{t}, & 0 < \frac{x}{t} < 1 \\ 1, & \frac{x}{t} > 1 \end{cases}$$

$$\Leftrightarrow \quad u(x+\tfrac{1}{3},t) = \begin{cases} 0, & x+\frac{1}{3} < 0 \\ \frac{x+\frac{1}{3}}{t}, & 0 < x+\frac{1}{3} < t \\ 1, & x+\frac{1}{3} > t \end{cases}$$

$$\Leftrightarrow \quad u(x+\tfrac{1}{3},t) = \begin{cases} 0, & x < -\frac{1}{3} \\ \frac{x+\frac{1}{3}}{t}, & -\frac{1}{3} < x < t-\frac{1}{3} \\ 1, & x > t-\frac{1}{3} \end{cases}$$

(*) ------

$\boxed{2}$ Now at $x = \frac{1}{3}$, let $X = x - \frac{1}{3}$.

The soln to the RP at $x = \frac{1}{3}$ becomes

(IC) $$\begin{array}{c} u(X,0) \\ \| \\ u(x-\frac{1}{3},0) \end{array} = \begin{cases} 1 \ (=u_L), & X < 0 \\ 0 \ (=u_R), & X > 0 \end{cases}$$

$\Rightarrow$ Again, $s = \dfrac{[f]}{[u]} = \dfrac{1}{2}$

$\Rightarrow$ Since $\lambda(u_L) > s > \lambda(u_R)$, it's a shock soln.

$$\begin{array}{ccccc} \lambda(u_L) & > & s & > & \lambda(u_R) \\ \| & & \| & & \| \\ u_L & & \frac{1}{2} & & 0 \\ \| & & & & \\ 1 & & & & \end{array}$$

$$\Rightarrow \quad u(X,t) = \begin{cases} 1 \; (=u_L), & \underset{=\, x-\frac{1}{3}}{X} < \frac{1}{2}t \\ 0 \; (=u_R), & \underset{=\, x-\frac{1}{3}}{X} > \frac{1}{2}t \end{cases}$$

$(t>0)$

$$\Leftrightarrow \quad u\left(x-\tfrac{1}{3},t\right) = \begin{cases} 1, & x < \frac{1}{2}t + \frac{1}{3} \\ 0, & x > \frac{1}{2}t + \frac{1}{3} \end{cases}$$

$(**)$

Finally, combining the two solutions $(*)$ & $(**)$:

$$u(x,t) = \begin{cases} 0, & x < -\frac{1}{3} \quad \boxed{A} \\ \dfrac{x+\frac{1}{3}}{t}, & -\frac{1}{3} < x < t-\frac{1}{3} \quad \boxed{B} \\ 1, & t-\frac{1}{3} < x < \frac{1}{2}t+\frac{1}{3} \quad \boxed{C} \\ 0, & x > \frac{t}{2}+\frac{1}{3} \quad \boxed{D} \end{cases}$$

Further, we find the two char. lines ① & ②:

Recall char lines:

$$x = \lambda(u(\xi,0))\,t + \xi$$

$\Rightarrow$ line ① : $\xi = -\frac{1}{3}$ and $\lambda(u(-\frac{1}{3},0))$
$= u(-\frac{1}{3},0)$
$= 1$

(i) $\boxed{x = t - \frac{1}{3}}$

$\Rightarrow$ line ② : Note this is NOT a char line, but a discont. (or shock) that travels with the speed $s = \frac{1}{2}$, emanating from $x = \frac{1}{3}$

(ii) $x = st + \frac{1}{3} = \frac{1}{2}t + \frac{1}{3}$

Lastly; $t \le \frac{4}{3}$ is obtained by considering line ① = line ②

$$\Rightarrow \quad t - \frac{1}{3} = \frac{t}{2} + \frac{1}{3}$$

$$\Rightarrow \quad \frac{t}{2} = \frac{2}{3} \quad \therefore \quad t = \frac{4}{3}$$