Things to know

ODE

① One-step method : FE, BE, Trap, RK

Multistep method : leapfrog, AB, AM

② $E_g$, $E_T$, $E_{step}$

convergent, consistent, stable, orders

Consistency + stability $\Longleftrightarrow$ Convergence

(ex) zero-stable (ODE)

Absolute stable (PDE)

High-order

③ RK : consistency condition ($\Leftrightarrow$ at least 1st order)

$\Leftrightarrow$ $E_g \to 0$, $dt \to 0$

④ LMM : char polys, $\rho$ & $\sigma$ for consistency condition

$$\begin{cases} \rho(1) = 0 \\ \rho'(1) = \sigma(1) \end{cases}$$

$\begin{cases} \text{All consistent } \underline{\text{one-step}} \to \text{convergent (Lipschitz on } \Psi) \\ \text{Consistent + } \underline{\text{zero-stable}} \to \text{convergent (LMM)} \end{cases}$

(ex) difference eqn to show $\exists$ ODE consistent but NOT convergent.
linear

→ zero-stability (root condition) using $\underline{\rho}$ ($\Delta t \to 0$)

① $|\xi_i| \le 1$ , $\xi_i$ : distinct

② $|\xi_i| \cancel{\le} 1$ , $\xi_i$ : repeated

→ Dahlquist Thm for ODE (linear & nonlinear)

→ zero-stability is enough for convergent for consistent method,

→ zero-stability is NOT enough to guarantee a soln converges to a correct soln,

→ absolute stability ($\Delta t > 0$),
- stability region, stability condition for $\Delta t > 0$,
- growth factor (amplification factor)
- stability poly $\pi = \rho(\xi) - z\sigma(\xi)$, root condition

⑤ Richardson's Extrapolation (one way to achieve high-order)

→ $\begin{cases} \Delta t,\ 2\Delta t,\ 4\Delta t, \cdots & \text{for implicit} \\ \Delta t,\ \Delta t/2,\ \Delta t/4, \cdots & \text{for explicit} \end{cases}$

global error estimator.

⑥ stiff ODE, why? How?

$\Delta t \le \Delta t_{stab} \ll \Delta t_{acc}$

⑦ BVP
- Shooting ✓
- FDM $\rightsquigarrow$ $A\mathbf{x}=b$, BC

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PDE ( orders, $E_{r}$, $O(\Delta t^r + \Delta x^p)$ )

① Parabolic PDEs : $u_t = k u_{xx}$ , $k>0$

→ FTCS, CS

→ MOL ; PDE based on ODE theory ✓

→ Stability theory PDE vs. ODE
- ODE: fixed in size (# of dep. vars) (zero-stab) $\Delta t \to 0$
- PDE: not fixed in size ( varies with $N$ )
  → abs. stab ; $\Delta t$ & $\Delta x$

CFL condition ; bound on

$$\begin{cases} \dfrac{k\Delta t}{\Delta x^2} & \text{for para} \\ a\dfrac{\Delta t}{\Delta x} & \text{for hyp.} \end{cases}$$

→ Von Neumann Analysis for stability $|e^{\alpha \Delta t}| \le 1$

$Ca = ?$, $\left( Ca = \dfrac{k\Delta t}{\Delta x^2} \right)$

(4) Hyperbolic PDE $u_t + a u_x = 0$ ($a > 0$ or $a < 0$)

- char lines (or curves)
- properties of $u$ along char. lines,
- FTCS fails, upwind good., LF, LW, leapfrog
  FTBS, FTFS
- $O(\Delta t^p + \Delta x^q)$, $E_T$
- Von Neumann stab. analysis $C_a = ?$
  ($C_a = \frac{a \Delta t}{\Delta x}$)
- Consistent, stability, CFL condition
- Modified Eqns
  → dissipation errors (1st order methods)
  → dispersion errors (2nd order methods)