AMS 213B Homework 6 – due Wednesday, May 25, 2016

Note: In this homework we are interested in solving the 1D linear scalar advection equation $u_t + au_x = 0$ with $a > 0$.

Simple Notation: $C_a \equiv \frac{a\Delta t}{\Delta x}$

Problem 1

Perform the local truncation analysis for the Lax-Friedrichs method to show the method is $\mathcal{O}(\Delta t + \Delta x^2)$. Please also show when the method becomes consistent.

Problem 2

Perform the local truncation analysis for the Lax-Wendroff method to show the method is $\mathcal{O}(\Delta t^2 + \Delta x^2)$. Please also show when the method becomes consistent.

Problem 3

Use the von Neumann stability analysis to show that the Lax-Wendroff method is stable for $|C_a| \leq 1$.

Problem 4

Use the von Neumann stability analysis to prove that the explicit Forward in Time Centered in Space (FTCS) for both $a > 0$ and $a < 0$

$$U_j^{n+1} = U_j^n - \frac{C_a}{2}\Big(U_{j+1}^n - U_{j-1}^n\Big) \tag{1}$$

is unconditionally unstable.

Problem 5

Use the von Neumann stability analysis to prove that the implicit Backward in Time Centered in Space (BTCS) for both $a > 0$ and $a < 0$

$$U_j^{n+1} = U_j^n - \frac{C_a}{2}\Big(U_{j+1}^{n+1} - U_{j-1}^{n+1}\Big) \tag{2}$$

is unconditionally stable.

Problem 6

Show that the modified equation for the Lax-Friedrichs method is an advection-diffusion equation

$$u_t + au_x = \kappa u_{xx}, \tag{3}$$

with

$$\kappa = \frac{\Delta x^2}{2\Delta t}\Big(1 - C_a^2\Big). \tag{4}$$

Use the fact that the diffusion coefficient needs to be always non-negative, $\kappa \geq 0$, to verify the CFL condition for the Lax-Friedrichs method $0 \leq C_a \leq 1$.

Problem 7

Show that the modified equation for the Lax-Wendroff method is an advection-dispersion equation

$$u_t + au_x = \mu u_{xxx}, \tag{5}$$

with

$$\mu = \frac{a\Delta x^2}{6}\Big(C_a^2 - 1\Big). \tag{6}$$