AMS 213B Homework 3 – due Wed, Apr 27, 2016

Problem 1

Recall that an ODE method is said to be *A-stable* if its region of absolute stability contains the entire left half-plane $\{z \in \mathbb{C} : Re(z) \le 0\}$. Show that the implicit backward Euler method

$$U^{n+1} = U^n + \Delta t f(t^{n+1}, U^{n+1}) \tag{1}$$

is A-stable when applied for solving the linear model equation $u'(t) = \lambda u(t)$, i.e., $f(t,u) = \lambda u(t)$, for $\lambda \in \mathbb{C}$.

Problem 2

Use the method of undetermined coefficients to derive an explicit two-step method of the form

$$U^{n+2} = \alpha U^{n+1} + \Delta t\Big[\beta_1 f(U^{n+1}) + \beta_2 f(U^n)\Big]. \tag{2}$$

That is to say, please determine α, β_1 and β_2.

Problem 3

Show that the two-step method

$$U^{n+2} = \frac{1}{2}(U^{n+1} + U^n) + \frac{\Delta t}{4}\Big[4f(U^{n+2}) - f(U^{n+1}) + 3f(U^n)\Big], \quad n \ge 1 \tag{3}$$

with $u'(t) = f(u)$ is second-order. Also find the leading term in the local truncation error.

Problem 4

Solve the third-order linear difference equation

$$U^{n+3} = U^{n+2} + c(U^{n+1} - U^n) \quad n \geq 0, \quad 0 < c < 1, \tag{4}$$

with U^0, U^1, U^2 given. What can you say about $\lim_{n\to\infty} U^n$?

Problem 5

Consider the two-step method

$$U^{n+2} = 2U^n - U^{n+1} + \frac{\Delta t}{2}\Big[5f(U^{n+1}) + f(U^n)\Big]. \tag{5}$$

Discuss the followings on the method: (i) order of accuracy, (ii) consistency, and (iii) zero-stability.

Furthermore, what can you say about convergence of the method? If you think the method is not convergent, please justify your assertion by applying the method to solve a simple IVP (i.e., please choose a simple form of $f(u)$), and by directly showing its numerical solution diverges as $n \to \infty$. On the other hand, prove your claim if you think the method is convergent.

Problem 6

Consider the nonhomogeneous IVP:

$$\begin{cases} u'(t) = f(u) + g(t), \\ u(0) = 1, \end{cases} \tag{6}$$

where $f(u) = -u$ and $g(t) = 2\cos t$. The exact solution is $u(t) = \cos t + \sin t$.

(a) Use the Forward Euler's method to derive a new second-order method using Richardson extrapolation. To obtain a new method, please proceed the following steps:

- Since the Forward Euler's method is explicit, you first need to find the absolute stability bound for Δt of the homogeneous equation. What is the maximum allowed bound, $\Delta t_{\max}$?

- After finding the bound, you consider obtaining a new formula by halving the timestep (i.e., $\Delta t/2$), instead of doubling it (i.e., $2\Delta t$) in order to avoid that $2\Delta t$ is outside the stability bound.
- The resulting formula based on Richardson extrapolation will be your new second-order scheme, $U_{\Delta t}^{n,(1)}$.

(b) Find a formula for the global error estimator of the method.

(c) Write a MATLAB program to fill out the following table:

t	$U_{\Delta t}^n$	$U_{\Delta t/2}^n$	$u(t^n)$	$U_{\Delta t}^{n,(1)}$	$u^n - U_{\Delta t}^n$	global error estimator
2						
4						
6						
8						
10						

Please note that $U^n = U_{\Delta t}^n$ in the above. Also note that you are advancing twice more steps for $U_{\Delta t/2}^n$ than for $U_{\Delta t}^n$. Please use two different time step sizes, $\Delta t = 0.5\Delta t_{\max}$ and $\Delta t = 0.25\Delta t_{\max}$.