AMS 213B Homework 2 – due Monday, Apr 18, 2016

Problem 1

Consider the IVP:

$$\begin{cases} u'(t) = f(t, u(t)), \\ u(0) = u^0. \end{cases} \tag{1}$$

Show that the following methods are consistent by using the **definition of consistency** (i.e., $E_{LT}^{n+1} \to 0$ as $\Delta t \to 0$):

(a) The second-order trapezoidal method (implicit):

$$U^{n+1} = U^n + \frac{\Delta t}{2}\Big[f(t^n, U^n) + f(t^{n+1}, U^{n+1})\Big] \tag{2}$$

(b) The second-order midpoint (or leapfrog) method (explicit):

$$U^{n+1} = U^{n-1} + 2\Delta t f(t^n, U^n) \tag{3}$$

(c) The second-order backward difference method (implicit):

$$U^{n+1} = \frac{1}{3}\Big[4U^n - U^{n-1} + 2\Delta t f(t^{n+1}, U^{n+1})\Big] \tag{4}$$

For (a), (b), and (c), you should be able to show orders of accuracy because you're using the definition of consistency. Please confirm orders of accuracy for them.

For the following methods show that they are consistent by using **the characteristic polynomials**:

(d) The Heun's method (a two-stage, second-order explicit Runge-Kutta method):

$$U_1 = U^n \tag{5}$$
$$U_2 = U^n + \Delta t f(t^n, U_1) \tag{6}$$
$$U^{n+1} = U^n + \frac{\Delta t}{2}\Big[f(t^n, U_1) + f(t^{n+1}, U_2)\Big] \tag{7}$$

(e) The three-step third-order Adams-Bashforth method (explicit):

$$U^{n+3} = U^{n+2} + \frac{\Delta t}{12}\Big[5f(t^n, U^n) - 16f(t^{n+1}, U^{n+1}) + 23f(t^{n+2}, U^{n+2})\Big] \tag{8}$$

(f) The three-step third-order Adams-Moulton method (implicit):

$$U^{n+3} = U^{n+2} + \frac{\Delta t}{24}\Big[f(t^n, U^n) - 5f(t^{n+1}, U^{n+1}) + 19f(t^{n+2}, U^{n+2}) + 9f(t^{n+3}, U^{n+3})\Big] \tag{9}$$

Problem 2

Find *all* second-order two-step formulas of the form

$$U^{n+2} + \alpha_1 U^{n+1} + \alpha_0 U^n = \Delta t\Big[\beta_0 f(U^n) + \beta_1 f(U^{n+1}) + \beta_2 f(U^{n+2})\Big]. \tag{10}$$

(Hint: Use only $E_{LT}^{n+2} = \mathcal{O}(\Delta t^2)$ to find three relationships among the five unknowns, $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$. You should be able to find that the first two conditions are the consistency conditions of LMM.) Please also find the formulas for explicit method.

Problem 3

Write a MATLAB program to solve the linear IVP given by

$$\begin{cases} u'(t) = f(t, u) = \lambda u(t), \\ u(0) = 1, \end{cases} \tag{11}$$

with two choices of λ

$$\lambda = \begin{cases} 1, \\ -1. \end{cases} \tag{12}$$

Note that the exact solution is given by $u(t) = e^{\lambda t}$. Take the temporal discretization given by

$$t^n = n\Delta t, \quad n = 1, 2, \cdots, \tag{13}$$

with the final time step $t_{\max} = 5$. Use five different grid resolutions $N = 5, 10, 20, 40$ and 80, from which you determine $\Delta t = (t_{\max} - t^0)/(N+1) = 5/(N+1)$. Implement the following numerical methods to solve the IVP:

(a) the first-order forward Euler method,
(b) the second-order Heun's method (see part (d) in Problem 1), and
(c) the third-order three-step Adams-Bashforth method (see part (e) in Problem

1).

Since self-starting is not available in the Adams-Bashforth method you need to use other methods to compute U^1 and U^2. In this homework you can readily use the forward Euler to compute U^1, and the second-order Heun's method to compute U^2. Once you have these two solutions you can use the Adams-Bashforth for U^n, $n \geq 3$.

Now, please do the followings:

(1) Produce total of 10 plots (i.e., five Δt values for two λ values), in which each plot has three numerical solutions plotted as functions of t, along with the exact solution.

(2) What do you find as you refine Δt for $\lambda = 1$? Can you identify the best and the worst numerical methods (or none) when you compare them with the exact solution? Please explain each of the numerical behavior mathematically.

(3) What do you find as you refine Δt for $\lambda = -1$? Can you identify the best and the worst numerical methods (or none) when you compare them with the exact solution? Please explain each of the numerical behavior mathematically.