§5. Spectral Methods

Book by Hesthaven, Gottlieb, & Gottlieb. [spectral methods for time-dep. problems]

II. Why spectral methods?

→ Spectral methods are global methods:

computation at any pt. depends both on information at

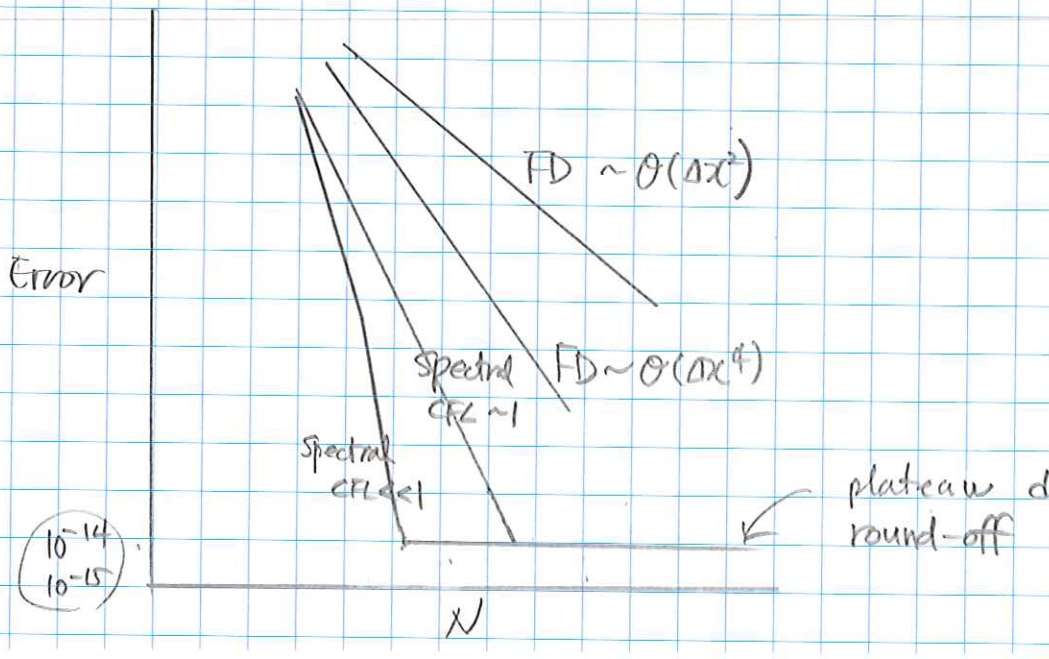
① neighboring pts, &

② the entire domain

→ Finite difference methods are local methods:

computation at any pt. only uses the local neighboring pts.

→ The key idea of using the global methods lies in the fact that the global scheme is superior in accuracy to the local scheme, with only using smaller number of grid pts.



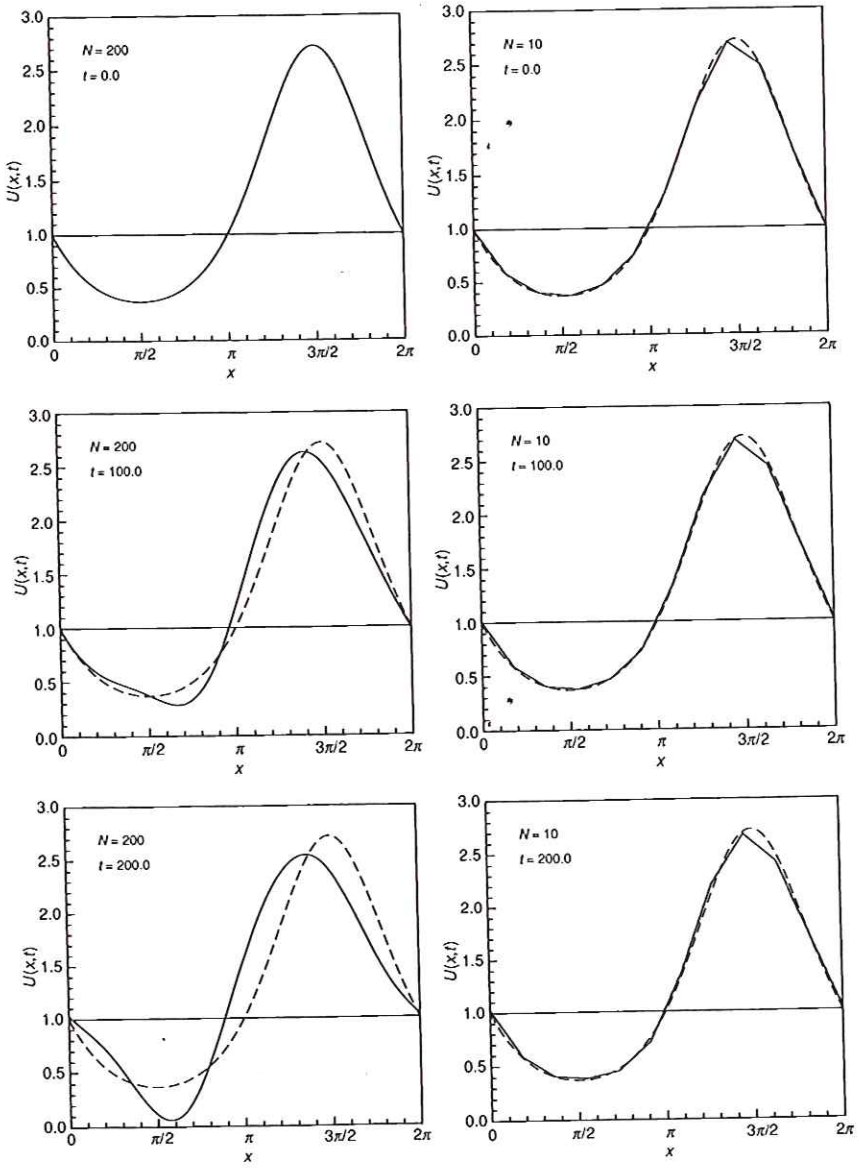


Figure 1.2 An illustration of the impact of using a global method for problems requiring long time integration. On the left we show the solution of Equation (1.1) computed using a second-order centered-difference scheme. On the right we show the same problem solved using a global method. The full line represents the computed solution, while the dashed line represents the exact solution.

→ The local methods are a good choice for slowly varying fns because the use of local polynomial interpolant based on a small # of interpolating grid pts is very reasonable.

Ex) hyperbolic PDEs (waves)

→ However, using low-degree local polys in the local methods to approximate sols containing very significant spatial or temporal variation requires a very fine grid pts in order to accurately resolve the fn.

→ And, the use of fine grids requires expensive computional resources in simulations.

→ A good alternative, in this sense, is the global methods, which is good for fast varying fns (e.g., elliptic & parabolic PDEs) approximations.

→ This motivates us to study spectral methods.

[2] Classification of spectral methods

→ Spectral methods are one of the "big three" technologies for numerical PDEs:

- 1950s: finite difference methods
- 1960s: finite element methods
- 1970s: spectral methods

→ the key components for formulating spectral methods:

(i) trial fns (basis fns, approximating fns, or expansion fns)
(ii) test fns (weight fns), where

(a) $u_N(x) = \sum_{k \in I_N} a_k(t)\, \phi_k(x)$

- $\phi_k(x)$: trial fns
- $a_k(t)$: expansion coeffs.
- $k \in I_N$: truncated expansion (finite)
- $u_N(x)$: approximate soln $\sim u(x)$, where $u(x)$ satisfies $Lu = f$.

where $u_N(x)$ is obtained by asking a condition:

(b) $\int_a^b [L u_N - f]\, \psi_k(x)\, dx = 0$: orthogonality

- $\psi_k(x)$: test fns
- $L u_N - f$: Residual ($\neq 0$)
- a, b: periodic or non-periodic

→ (I) trial fns: provide the approximate representation of the soln. (⇒ (a))

(II) test fns: ensure that the differential eqn $Lu=f$ and BCs are satisfied as closely as possible by the truncated series expansion, by minimizing the residual (b) which is produced by the truncated expansion instead of the exact soln.

→ Spectral methods can be viewed as "the method of weighted residuals"

Classifications

(1) The choice of trial fns $\phi_k(x)$:

→ distinguishes the classical spectral methods from FE & FD (i.e, global vs. local)

→ Fourier spectral vs. polynomial spectral

① Fourier spectral: periodic domain

(ex) $\phi_k(x) = e^{ikx}$

② poly. spectral: non-periodic domain

(ex) $\phi_k(x)$ = Chevyshev poly $T_k(x)$.

(2) the choice of test fns $\psi_k(x)$:

→ distinguishes between the three earliest types of spectral schemes:

① Galerkin: $\psi_k(x) = \phi_k(x)$

→ same as the trial fn

② Collocation: $\psi_k(x) = \delta(x - x_k)$

→ Dirac-delta fns (distributions)

→ also called the "pseudospectral"

③ Tau method: similar to Galerkin in the way to enforce (b), but ψ_k do not need to satisfy BCs.

→ supplementary set of Eqns is needed to apply BCs

[3] Examples

Def: The truncated Fourier series

$$\mathcal{S}_N u(x) = \sum_{|n| \le N/2} \hat{u}_n e^{inx}, \quad \hat{u}_n = \frac{1}{2\pi}\int_0^{2\pi} u(x) e^{-inx} dx.$$

is a projection to the finite dimensional space

$$\hat{B}_N = \text{span}\left\{ e^{inx} \;\middle|\; |n| \le \frac{N}{2} \right\},$$

$$\dim(\hat{B}_N) = N+1.$$

Note that $\hat{B}_N$ is the space of trigonometric polys of order N.

§3.1. Fourier-Galerkin method:

$$\begin{cases} \phi_k(x) = e^{ikx} \\ \psi_k(x) = \frac{1}{2\pi} e^{-ikx} \quad \text{(essentially the same as } \phi_k(x)\text{)} \\ [0, 2\pi] \text{ : periodic} \end{cases}$$

→ Consider

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \mathcal{L} u(x,t), & x \in [0,2\pi], \; t \ge 0 \\ u(x,0) = g(x), & x \in [0,2\pi], \; t = 0 \end{cases}$$

→ We seek solns $u_N(x,t) \in \hat{B}_N$:

$$u_N(x,t) = \sum_{|n| \le \frac{N}{2}} a_n(t)\, e^{inx},$$

→ truncated, ① NOT exact. --- This is (a).

→ $a_n(t)$: unknown coeffs, which will be determined by the method, ($a_n \neq \hat{u}_n$)

→ $a_n(t)$ are determined by the requirement that the residual

$$R_N(x,t) = \frac{\partial u_N(x,t)}{\partial t} - \mathcal{L} u_N(x,t)$$

is orthogonal to $\hat{B}_N$ (i.e, (b))

→ If we express R_N in terms of the Fourier series,

$$R_N(x,t) = \sum_{|n| \le \infty} \hat{R}_t(t)\, e^{inx},$$

then the orthogonality requirement yields:

$$\boxed{\hat{R}_N(t) = \frac{1}{2\pi}\int_0^{2\pi} R_N(x,t)\, e^{-inx}\, dx = 0}, \quad \forall\, |n| \le \frac{N}{2}.$$

---①

→ ∃ (N+1) ODEs to determine the (N+1) unknowns, $a_n(t)$, and the corresponding ICs are:

$$u_N(x,0) = \sum_{|n| \le \frac{N}{2}} a_n(0)\, e^{inx}, \text{ where}$$

$$a_n(0) = \frac{1}{2\pi}\int_0^{2\pi} g(x)\, e^{-inx}\, dx \quad (\leftarrow \text{given})$$

→ The method is defined by the requirement ① that the orthogonal projection of the residual onto the space $\hat{B}_N$ is zero.

Rmk: Fourier Galerkin method is

(i) very efficient for linear, constant coeff. problems

(ii) but it becomes very complicated for variable coffiients and nonlinear problems.

(iii) the main drawback is the need to derive and solve a different system of ODEs for each problem.

§3.2. Fourier – Collocation method

$$\begin{cases} \varphi_k(x) = e^{ikx} \\ \psi_k(x) = \delta(x - x_k) \\ [0, 2\pi] \ ; \text{ periodic} \end{cases}$$

→ Also called the pseudospectral method,

→ Instead of requiring ① using $\psi_n(x) = \frac{1}{2\pi} e^{-inx}$, we use $\psi_n(x) = \delta(x - x_n)$; then we obtain

$$0 = \int_0^{2\pi} R_N(x,t)\, \delta(x - x_n)\, dx = R_N(x_n) \quad \cdots \ ②$$

→ ② is requiring that the residual vanishes at the collocation pts x_n, $n = 0, 1, \cdots, N-1$,

→ This removes the need to evaluate the integral (or the inner product) which caused such difficulty in the Fourier – Galerkin method.

→ ② yields N eqns to determine the N pt. values, $u_N(x_n, t)$, of the numerical soln.

→ In other words, the pseudospectral appn $u_N(x,t)$ satisfies

$$\frac{\partial u_N(x,t)}{\partial t} - I_N \mathcal{L} u_N(x,t) = 0, \text{ where}$$

I_N is the interpolating operator.

§3.3. Tau methods.

→ For non-periodic BCs, we cannot use the Fourier-Galerkin or Fourier-Collocation (pseudospectral) methods.

→ Alternates are to use the polynomial spectral methods (the trial fns are now polys)

$\phi_n(x)$ = Legendre, or Chebyshev.

→ The choice of the test fns distinguishes:

(1) Galerkin

(2) collocation

(3) Tau

→ Since we've looked at the examples of (1) & (2) in the Fourier spectral method in periodic BCs, let's take a look at (3) Tau method in non-periodic case now.

→ Consider

$$\frac{\partial u(x,t)}{\partial t} = \mathcal{L}u(x,t), \quad x \in [a,b], \ t \geq 0$$

$$B_L u = 0, \quad t \geq 0$$

$$B_R u = 0, \quad t \geq 0$$

$$u(x,0) = f(x), \quad x \in [a,b], \ t = 0,$$

B_L & B_R : the boundary operators at $x=a$ & $x=b$, resp.

Def. $B_N = \text{span}\{\underbrace{\phi_n(x)}_{\text{a poly (trial fun)}} \in \text{span}\{x^k \mid k=0,\cdots,n\} \mid B_L\phi_n = B_R\phi_n = 0,\ n=0,\cdots,N\}$

→ The Tau method seeks soln $u_N(x,t) \in B_N$, without projecting the residual onto the space B_N but rather onto the poly. space

$$P_{N-k} = \text{span}\{x^n \mid n=0,\cdots,N-k\},$$

where k = # of BCs.

→ the approximant is a poly of degree N, but k degrees of freedom are used to enforce the BCs.

→ The test fcns do not satisfy BCs.

→ Consider
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} \text{ on } [-1,1] \\ \text{BC: } u(1,t) = h(t) \\ \text{IC: } u(x,0) = f(x). \end{cases}$$
for simplicity, we only consider one BC at $x=1$. ($k=1$ then)

→ If choosing the Legendre tau method (i.e, the trial fcns $\phi_n(x) = P_n(x)$: the Legendre poly, defined by

$$\frac{d}{dx}(1-x^2)\frac{dP_n(x)}{dx} + n(n+1)P_n(x) = 0, \text{ or equivalently,}$$

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k}\binom{2n-2k}{n} x^{n-2k}, \quad [n/2]: \text{nearest integer.}$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

→ We seek solns $u_N \in B_N$ of the form

$$u_N(x,t) = \sum_{n=0}^{N} a_n(t) P_n(x),$$

with the additional constraint that

$$\sum_{n=0}^{N} a_n(t) P_n(1) = \sum_{n=0}^{N} a_n(t) = h(t)$$

to ensure that the soln satisfies the BC.

→ The first N eqns are found by requiring that

$$R_N(x,t) = \frac{\partial u_N}{\partial t} - \frac{\partial u_N}{\partial x}$$

is L^2-orthogonal to P_{N-1}:

$$\frac{2k+1}{2} \int_{-1}^{1} R_N(x,t) P_k(x) \, dx = 0, \quad k = 0, 1, \cdots, N-1. \quad \text{--- (2)}$$

(→ Legendre expansion)

→ This yields N ODEs:

$$\frac{da_n}{dt} = (2n+1) \sum_{\substack{p=n+1 \\ p+n: \text{odd}}}^{N} a_p(t), \quad n = 0, \cdots, N-1.$$

→ The remaining task is to solve the ODEs to find a_n.

Rmk. Since we project the residual R_N onto P_{N-k}, rather than B_N, the mass matrix remains diagonal due to the orthogonality of the Legendre polys, and the stiffness matrix is obtained directly from the properties of the poly. basis, typically resulting in schemes much simpler than the Galerkin appn.