§4. Elliptic PDEs

I Steady-state ($\partial/\partial t = 0$) Diffusion

→ One can often consider the characters given by elliptic PDEs;

$$\boxed{\nabla^2 u = 0} \quad \cdots \quad ①$$

as the steady-state problem of the diffusion equations given by parabolic PDEs;

$$\boxed{\frac{\partial u}{\partial t} = \nabla^2 u} \quad \cdots \quad ②$$

→ If we consider ① in 2D, we get,

$$\boxed{\nabla^2 u = u_{xx} + u_{yy}}$$

→ Writing ② in a more general form, together with

(i) $f(x,y,t)$; a source term,

(ii) $k(x,y) > 0$; a diffusion coeff.

$$\boxed{u_t = \nabla \cdot (k \nabla u) - f} \quad \cdots ③ \quad \text{on } \Omega \subset \mathbb{R}^2,$$

$$\Leftrightarrow \quad u_t = (k u_x)_x + (k u_y)_y - f$$

→ In ③, the soln. $u(x,y,t)$ will vary both in time & space.

→ Together with ③, we also need IC & BC;

$$\begin{cases} \text{(i) IC: } & u(x,y,0) \text{ in } \Omega \\ \text{(ii) BC: } & u(x,y,t), \ \forall (x,y) \in \partial\Omega, \ \forall t > 0. \end{cases}$$

→ We expect a steady-state ($\partial/\partial t = 0$) eqn if

(i) BC is time-independent; i.e,

$$u(x,y,t) = u(x,y), \quad \forall (x,y) \in \partial\Omega,$$

(ii) the source term is time-independent; i.e,

$$f(x,y,t) = f(x,y),$$

and hence arriving to get:

$$\boxed{(ku_x)_x + (ku_y)_y = f} \quad \cdots \text{④}$$

→ We will consider our target model PDE for elliptic takes of the form

(i) $\boxed{(ku_x)_x + (ku_y)_y = f}$, $\quad k = k(x,y) > 0$,

or

(ii) $\boxed{k u_{xx} + k u_{yy} = f}$, if $k = \text{constant} > 0$.

Ex. If $k=1$, we have the Poisson Eqn:

$$\boxed{u_{xx} + u_{yy} = f}, \qquad f \not\equiv 0.$$

Ex. If $k=1$ & $f \equiv 0$, then we get the Laplace Eqn:

$$\boxed{u_{xx} + u_{yy} = 0}$$

Ex. We need to specify BCs on $\partial\Omega$. For example,

Dirichlet BC: (BC on u itself)

$$u(x,y) = u_{BC}(x,y), \quad \forall (x,y) \in \partial\Omega$$

Neumann BC: (BC on $\partial u/\partial x$ & $\partial u/\partial y$)

$$\begin{cases} \dfrac{\partial u}{\partial x} = \alpha(x,y), \\ \dfrac{\partial u}{\partial y} = \beta(x,y), \quad \forall (x,y) \in \partial\Omega. \end{cases}$$

Mixed BC: Dirichlet + Neumann on $\partial\Omega$.

Rmk Why are we considering the elliptic eqns in 2D?

→ In 1D, the Laplace eqn : $u_{xx}(x) = 0$ on $[x_a, x_b]$.

then $u(x)$ is <u>trivial</u> which is a linear ftn

connecting $[x_a, u(x_a)]$ & $[x_b, u(x_b)]$.

BCs.

→ This is NOT true in 2D, and the solns to the Laplace eqn are called "<u>harmonic ftns</u>."

Rmk ① ∇^2 is called the Laplacian

② In mathematics, Δ is also used for the Laplacian, but we do not use Δ in order to avoid any confusion for Δx, Δy, Δz, Δt, etc.

2| The 5-pt. stencil for $\nabla^2 u$

→ Consider $k=1$ and hence

$$\boxed{u_{xx} + u_{yy} = f} \quad \cdots \text{①}.$$

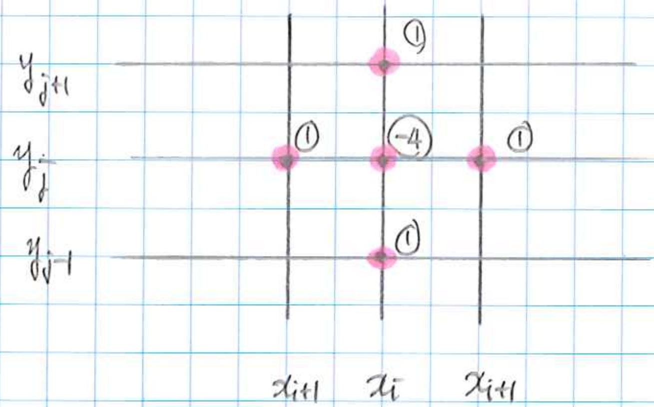
for our model eqn.

→ Let $\Omega = [0,1] \times [0,1]$, with Dirichlet BCs.

The grid is discretized using:

$$(x_i, y_j) = (i\Delta x, j\Delta y), \quad i,j = 1, \cdots, m.$$

(Note we do not use the cell-centered configuration any more).



← 5-pt stencil

→ Discrete form for ①:

$$\boxed{\frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{\Delta x^2} + \frac{U_{i,j-1} - 2U_{ij} + U_{i,j+1}}{\Delta y^2} = f_{ij}} \quad \cdots \text{②}$$

→ Assuming $\Delta x = \Delta y = h$, ② becomes

$$\boxed{\frac{1}{h^2}\left[U_{i,j-1} + U_{i-1,j} - 4U_{ij} + U_{i+1,j} + U_{i,j+1}\right] = f_{ij}} \quad \cdots \text{③}$$

3 Ordering the unknowns & eqns

(1) Natural rowwise ordering: We rewrite ③ into

$$\boxed{U_{i,j-1} + (U_{i-1,j} - 4U_{ij} + U_{i+1,j}) + U_{i,j+1} = h^2 f_{ij}} \quad \cdots ④$$

and we consider ④ as the $(i,j)^{th}$ element of the operation;

$$\underbrace{\begin{bmatrix} U_{1,j-1} \\ U_{2,j-1} \\ \vdots \\ U_{i,j-1} \\ \vdots \\ U_{m,j-1} \end{bmatrix}_{m\times 1}}_{5a} + \underbrace{\begin{bmatrix} -4 & 1 & & & & 0 \\ 1 & -4 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & -4 & 1 & \\ & & & 1 & -4 & 1 \\ 0 & & & & 1 & -4 \end{bmatrix}_{m\times m}}_{5b} \underbrace{\begin{bmatrix} U_{1,j} \\ U_{2,j} \\ \vdots \\ U_{ij} \\ \vdots \\ U_{mj} \end{bmatrix}_{m\times 1}}_{5c} + \underbrace{\begin{bmatrix} U_{1,j+1} \\ U_{2,j+1} \\ \vdots \\ U_{i,j+1} \\ \vdots \\ U_{m,j+1} \end{bmatrix}_{m\times 1}}_{5d} = h^2 \underbrace{\begin{bmatrix} f_{1j} \\ f_{2j} \\ \vdots \\ f_{ij} \\ \vdots \\ f_{mj} \end{bmatrix}_{m\times 1}}_{5e} \quad \cdots ⑤$$

→ Let's introduce compact notations;

(i) $U^{[j]} = \begin{bmatrix} U_{1j} \\ U_{2j} \\ \vdots \\ U_{ij} \\ \vdots \\ U_{mj} \end{bmatrix}_{m\times 1}$, $f^{[j]} = \begin{bmatrix} f_{1j} \\ f_{2j} \\ \vdots \\ f_{ij} \\ \vdots \\ f_{mj} \end{bmatrix}_{m\times 1}$, $j = 1, \cdots, m$

(ii) $T = (5b)$, $m\times m$ matrix,

then we see that, using the identity matrix $I_{m\times m}$,

(iii) $(5a) = I\,U^{[j-1]}$

(iv) $(5b) * (5c) = T\,U^{[j]}$, & (vi) $(5e) = h^2 I f^{[j]}$.

(v) $(5d) = I\,U^{[j+1]}$

→ ⑤ can be cast into a compact notation:

$$\boxed{I\,U^{[j-1]} + T\,U^{[j]} + I\,U^{[j+1]} = h^2 I f^{[j]}}, \quad \cdots ⑥$$

for $j = 1, \cdots, m$.

→ Furthermore, we can write ⑥ into an even more compact form:

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} T & I & & & 0 \\ I & T & I & & \\ & \ddots & \ddots & \ddots & \\ 0 & & T & T & I \\ & & & I & T \end{bmatrix}}_{\equiv \underset{\sim}{A}} \underbrace{\begin{bmatrix} U^{[1]} \\ U^{[2]} \\ \vdots \\ U^{[m]} \end{bmatrix}}_{\equiv \underset{\sim}{U}} = \underbrace{\begin{bmatrix} f^{[1]} \\ f^{[2]} \\ \vdots \\ f^{[m]} \end{bmatrix}}_{\equiv \underset{\sim}{f}}, \quad \text{or}$$

$$\boxed{\underset{\sim}{A}\,\underset{\sim}{U} = \underset{\sim}{f}}$$

→ Note here $\underset{\sim}{A}$ is an <u>$m \times m$ block tridiagonal matrix</u> in which each block T or I is itself an $m \times m$ matrix.

Rmk Properties of $\underset{\sim}{A}$.

(i) very sparse,

(ii) $m^2 \times m^2$ matrix,

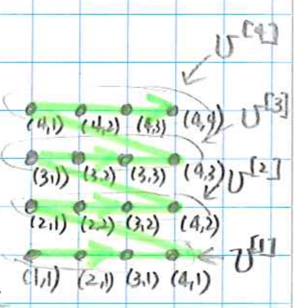
(iii) each of its block is $m \times m$ matrix,

(iv) each row of $\underset{\sim}{A}$ has at most 5 nonzeros, and at least m^2-5 elements that are zero.

(v) the 1 values in the I matrices are separated from the diagonal by $m-1$ zeros.

$$\underbrace{\begin{bmatrix}
\begin{bmatrix} -4 & 1 & & \\ 1 & -4 & 1 & \\ & 1 & -4 & 1 \\ & & 1 & -4 \end{bmatrix} & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} & & \\
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} & \begin{bmatrix} -4 & 1 & & \\ 1 & -4 & 1 & \\ & 1 & -4 & 1 \\ & & 1 & -4 \end{bmatrix} & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} & \\
 & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} & \begin{bmatrix} -4 & 1 & & \\ 1 & -4 & 1 & \\ & 1 & -4 & 1 \\ & & 1 & -4 \end{bmatrix} & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
 & & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} & \begin{bmatrix} -4 & 1 & & \\ 1 & -4 & 1 & \\ & 1 & -4 & 1 \\ & & 1 & -4 \end{bmatrix}
\end{bmatrix}}_{4^2 \times 4^2} = \underset{\sim}{A}_{4^2 \times 4^2}$$

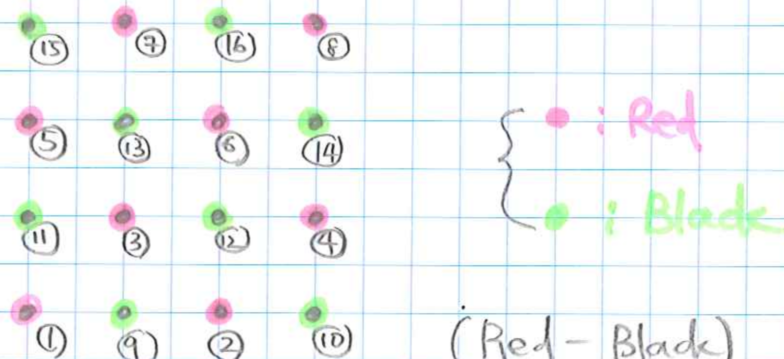
↓ the natural rowwise ordering of unknowns and eqns on 4×4 grid. (green: ordering)



(vi) This is called <u>the rowwise ordering</u> to construct $\underset{\sim}{A}$, since $U^{[j]}$ is a vector whose elements are the <u>j^{th} row</u> of the stencil.

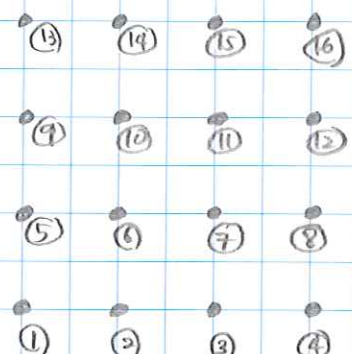
(2) <u>Red-Black ordering</u> (or <u>checkerboard ordering</u>)

→ The red-black ordering takes a configuration of the ordering;



<u>(Red-Black)</u>

this is in contrast to the rowwise ordering:



<u>(Rowwise)</u>

→ The red-black ordering is better because

(i) the all four neighbours of each red (or black) grid pt. are all black (or red),

(ii) hence, the matrix eqn has the structure:

$$\underbrace{\begin{bmatrix} D & H \\ H^T & D \end{bmatrix}_{m^2\times m^2}}_{\underline{A}} \underbrace{\begin{bmatrix} U_{Red} \\ U_{Black} \end{bmatrix}_{m^2\times 1}}_{\underline{U}} = \underbrace{\begin{bmatrix} f_{Red} \\ -f_{Black} \end{bmatrix}_{m^2\times 1}}_{\underline{f}}, \quad (\text{or } \underline{A}\,\underline{U} = \underline{f})$$

where

$$\begin{cases} D = -\dfrac{4}{h^2} I \; : \text{ a diagonal matrix, } \tfrac{1}{2}(m \times m) \\[2em] H = \text{a banded matrix, } \tfrac{1}{2}(m \times m), \text{ with four nonzeros around diagonal} \end{cases}$$

→ This is to re-group (3) into ; assuming U_{ij} is at (6) (Red)

$$\underbrace{\begin{bmatrix} -4 U_{ij}^{(6)} \end{bmatrix}}_{D \text{ (Red)}} + \underbrace{\begin{bmatrix} & U_{ij+1}^{(16)} & \\ U_{i-1,j}^{(13)} & + & U_{i+1,j}^{(14)} \\ & U_{ij-1}^{(12)} & \end{bmatrix}}_{H} \text{ (Black)} = h^2 f_{ij}$$

$$\to \overset{Dh^2}{\begin{bmatrix} -4 &&&&&&& \\ & -4 &&&&&& \\ && -4 &&&&& \\ &&& -4 &&&& \\ &&&& -4 &&& \\ &&&&& -4 && \\ &&&&&& -4 & \\ &&&&&&& -4 \end{bmatrix}}\overset{U_{Red}}{\begin{bmatrix} U_{(1)} \\ U_{(2)} \\ U_{(3)} \\ U_{(4)} \\ U_{(5)} \\ U_{(6)} \\ U_{(7)} \\ U_{(8)} \end{bmatrix}} + \overset{Hh^2}{\begin{bmatrix} 1 & 0 & 1 &&&&& \\ 1 & 1 & 0 & 1 &&&& \\ 1 & 1 & 1 & 0 & 1 &&& \\ & 1 & 1 & 1 & 0 & 1 && \\ && 1 & 1 & 1 & 0 & 1 & \\ &&& 1 & 1 & 1 & 0 & 1 \\ &&&& 1 & 1 & 1 & 0 \\ &&&&& 1 & 1 & 1 \end{bmatrix}}\overset{U_{Black}}{\begin{bmatrix} U_{(9)} \\ U_{(10)} \\ U_{(11)} \\ \boxed{U_{(12)}} \\ \boxed{U_{(13)}} \\ U_{(14)} \\ U_{(15)} \\ \boxed{U_{(16)}} \end{bmatrix}} = h^2 \overset{f_{Red}}{\begin{bmatrix} f_{(1)} \\ f_{(2)} \\ f_{(3)} \\ f_{(4)} \\ f_{(5)} \\ f_{(6)} \\ f_{(7)} \\ f_{(8)} \end{bmatrix}}$$

(1) (2) (3) (4) (5) (6) (7) (8)   (9) (10) (11) (12) (13) (14) (15) (16)

$$\Leftrightarrow \boxed{D\, U_{Red} + H\, U_{Black} = f_{Red}}$$

→ Similarly, if U_{ij} is at any black pt, we get

$$\boxed{H^T U_{Red} + D\, U_{Black} = -f_{Black}}, \quad \text{i.e.,}$$

$$
\underbrace{\begin{array}{cccccccc}
\multicolumn{8}{c}{\begin{bmatrix}
1 & 1 & 1 & & & & & \\
0 & 1 & 1 & 1 & & & & \\
1 & 0 & 1 & 1 & 1 & & & \\
 & 1 & 0 & 1 & 1 & 1 & & \\
 & & 1 & 0 & 1 & 1 & 1 & \\
 & & & 1 & 0 & 1 & 1 & 1 \\
 & & & & 1 & 0 & 1 & 1 \\
 & & & & & 1 & 0 & 1
\end{bmatrix}} \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}}_{H^T h^2}
\underbrace{\begin{bmatrix} U_{1} \\ U_{2} \\ U_{3} \\ U_{4} \\ U_{5} \\ U_{6} \\ U_{7} \\ U_{8} \end{bmatrix}}_{U_{Red}}
+
\underbrace{\begin{array}{cccccccc}
\multicolumn{8}{c}{\begin{bmatrix}
-4 & & & & & & & \\
 & -4 & & & & & & \\
 & & -4 & & & & & \\
 & & & -4 & & & & \\
 & & & & -4 & & & \\
 & & & & & -4 & & \\
 & & & & & & -4 & \\
 & & & & & & & -4
\end{bmatrix}} \\
9 & 10 & 11 & 12 & 13 & 14 & 15 & 16
\end{array}}_{D h^2}
\underbrace{\begin{bmatrix} U_{9} \\ U_{10} \\ U_{11} \\ U_{12} \\ U_{13} \\ U_{14} \\ U_{15} \\ U_{16} \end{bmatrix}}_{U_{Black}}
= -h^2
\underbrace{\begin{bmatrix} f_{9} \\ f_{10} \\ f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \\ f_{15} \\ f_{16} \end{bmatrix}}_{f_{Black}}
$$

for instance, if U_{ij} is at 13 ;

$$-\frac{1}{h^2}\left[-4U_{13} + U_{3} + U_{5} + U_{6} + U_{7}\right] = -f_{13}$$

$$\Leftrightarrow \frac{1}{h^2}\left[-4U_{13} + U_{3} + U_{5} + U_{6} + U_{7}\right] = f_{13}$$

→ Of course, in this example, we haven't included the BCs into the matrix $\underset{\sim}{A}$ (i.e., in $\underset{\sim}{D}$ & $\underset{\sim}{H}$), and hence the simple illustration shown here is not available near the boundary pts.