[5] The CFL (Courant-Lewy-Friedrichs) Condition

Def. Let $\begin{cases} \mathcal{D}_{Num} = \text{numerical domain of dependence}, \\ \mathcal{D}_{exact} = \text{analytical domain of dependence of the PDE.} \end{cases}$

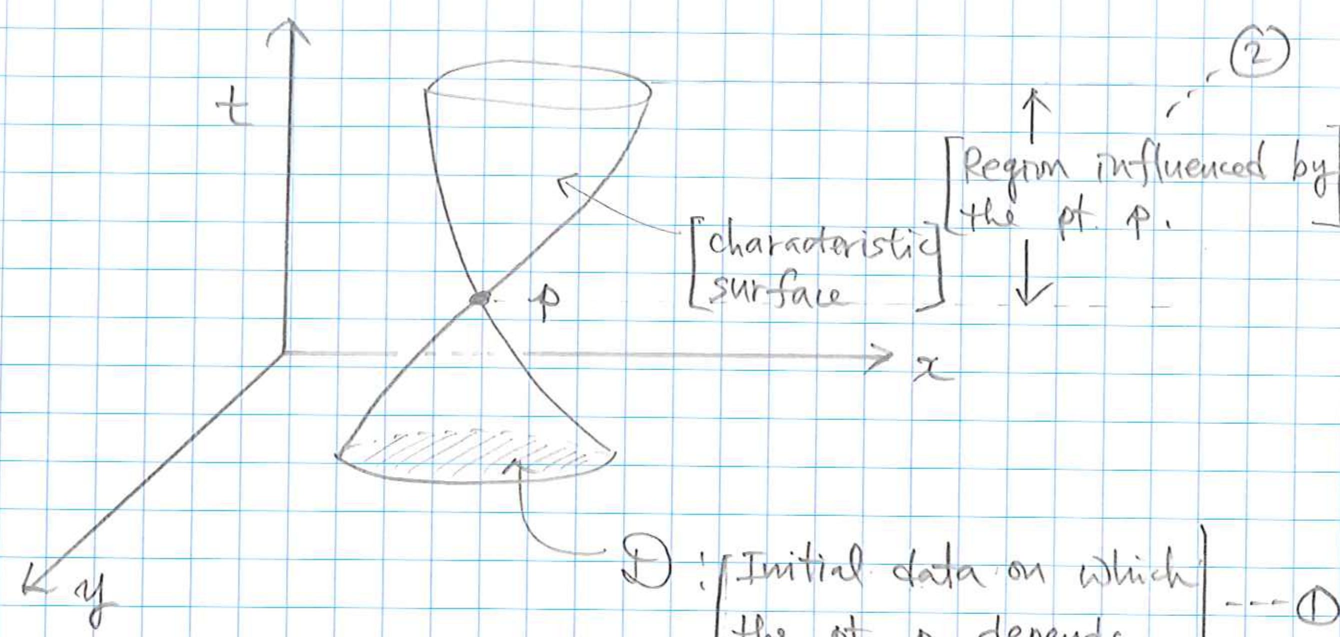
A numerical method can be convergent only if

$$\mathcal{D}_{Num} \supset \mathcal{D}_{exact}.$$

This is called the CFL condition.

Rmk. Recall that, in hyperbolic PDEs, we have;

(i) domain of dependence, &
(ii) range of influence.



$\Rightarrow$ $\begin{cases} \text{① is the domain of dependence of } p. \\ \text{② is the region of influence of } p. \end{cases}$

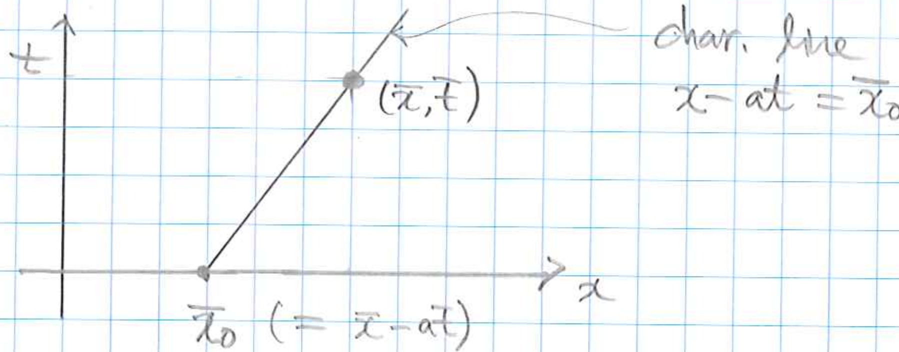
Rmk. Mathematically, we state them as follows:

The soln $u(x,t)$ at any pt. $(\bar{x},\bar{t})$ depends only on the initial data u^0 at a single pt., namely, $\bar{x}_0$ s.t. $(\bar{x},\bar{t})$ lies on the characteristic lines (or curves) through $\bar{x}_0$.

Case 1: linear scalar constant coeff. $(u_t + a u_x = 0)$

$\rightarrow$ $\exists$ a single characteristic line, defined by

$$x - at = \bar{x}_0.$$



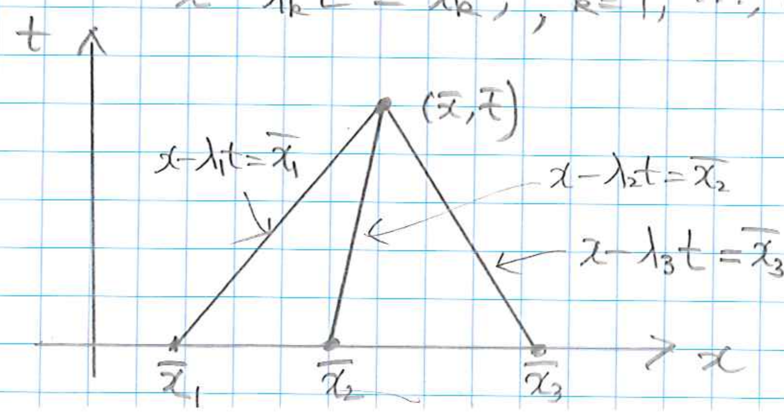
$\rightarrow$ $\mathcal{D} = \{\bar{x}_0\}$: domain of dep. of $(\bar{x},\bar{t})$.

Case 2: system of N advection Eqns. $(\underset{N\times 1}{\underset{\sim}{U_t}} + \underset{N\times N}{\underset{\sim}{A}} \underset{N\times 1}{\underset{\sim}{U_x}} = 0)$

$\rightarrow$ $\exists$ $\lambda_1, \cdots, \lambda_N$: multiple eigenvalues of $\underset{\sim}{A}$

$\rightarrow$ $\exists$ N many char. lines defined by

$$x - \lambda_k t = \bar{x}_k, \quad k = 1, \cdots, N$$

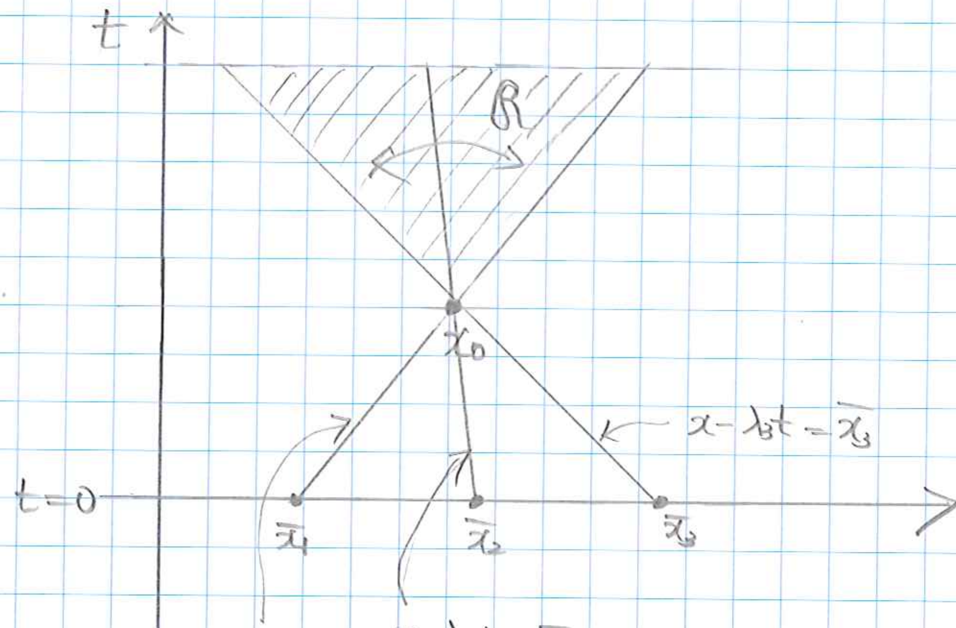


$\rightarrow$ $\mathcal{D} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$
$\quad = \{\bar{x} - \lambda_k \bar{t} \mid k = 1, \cdots, N\}$

On the other hand, the range of influence is defined by

$$\mathcal{R} = \{ (x,t) \mid \lambda_N t \le (x - x_0) \le \lambda_1 t \}, \text{ where}$$

$$\begin{cases} \lambda_1 = \max\limits_k \lambda_k, \\ \lambda_N = \min\limits_k \lambda_k. \end{cases}$$



Rmk. For $u_t = k u_{xx}$, $\mathcal{D} = \mathcal{R} = (-\infty, \infty)$

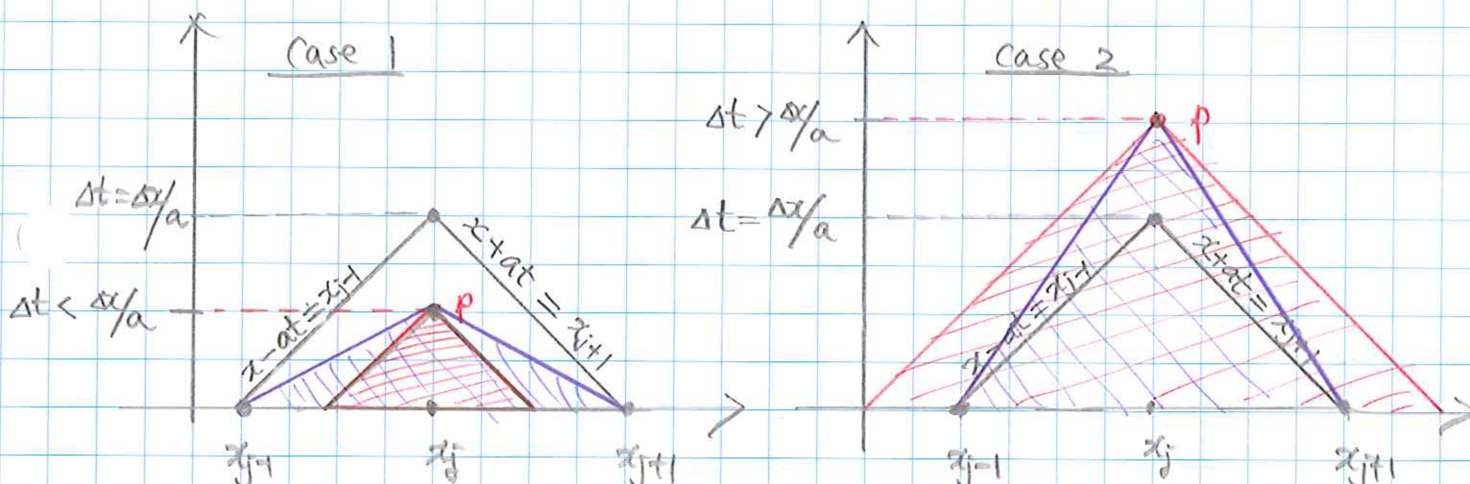
Ex. let's consider $u_t + a u_x = 0, \quad a > 0$, solved by the upwind method;

$$v_j^{n+1} = v_j^n - \frac{a\Delta t}{\Delta x}\left[v_j^n - v_{j-1}^n\right].$$

Consider two cases;

case 1: $\frac{a\Delta t}{\Delta x} < 1$, or $a\Delta t < \Delta x$

Case 2: $\frac{a\Delta t}{\Delta x} > 1$, or $a\Delta t > \Delta x$



→ The red triangles are the analytical domain of dependence of the pt. P.

→ The blue triangles are the numerical domain of dependence of the pt. P, using the upwind scheme.

→ Case 1;

$$D_{Num} \supset D_{exact}$$

$\therefore$ CFL condition ✓

Case 2;

$$D_{num} \subset D_{exact}$$

$\therefore$ CFL condition ✗

Conclusion:

We want (i) $\frac{a\Delta t}{\Delta x} \leq 1$ for stability, &

(ii) $\frac{a\Delta t}{\Delta x} \approx 1$ for accuracy.

Rmk. We often use $C_a = \begin{cases} \frac{a\Delta t}{\Delta x} & \text{for advection } (u_t + a u_x = 0) \ (a>0) \\ \frac{k\Delta t}{\Delta x^2} & \text{for diffusion } (u_t = k u_{xx}) \ (k>0) \end{cases}$

$\Rightarrow \boxed{0 < C_a \leq 1}$

Rmk $\Delta t = \begin{cases} \frac{C_a \Delta x}{|a|} \cdots ③ & \text{for advection } (a>0 \text{ or } a<0) \\ \frac{C_a \Delta x^2}{k} \cdots ④ & \text{for diffusion } (k>0) \end{cases}$

$$\Rightarrow \begin{cases} \Delta t_{adv} \equiv \frac{C_a \Delta x}{|a|} \\ \Delta t_{diff} \equiv \frac{C_a \Delta x^2}{k} \end{cases}$$

$\Rightarrow$ If we solve an advection–diffusion eqn;

$u_t + a u_x = k u_{xx}$, then

$\Delta t = \min \{ \Delta t_{adv}, \Delta t_{diff} \}$ for explicit schemes.

<u>Rmk</u> In general,

$$\Delta t_{adv} \gg \Delta t_{diff}, \quad \text{since} \quad \begin{cases} \Delta t_{adv} \sim O(\Delta x) \\ \Delta t_{diff} \sim O(\Delta x^2), \end{cases}$$

In this case, we want

$$\begin{cases} \text{explicit scheme for advection, \&} \\ \text{implicit scheme for diffusion,} \end{cases}$$

in order to overcome the small Δt_{diff} that makes computation very expensive.