[3] Von Neumann Stability Analysis

$\rightarrow$ As in the parabolic PDE case, we assume

$$\boxed{v_j^n = e^{\alpha n \Delta t} e^{I \xi j \Delta x}}, \quad I = \sqrt{-1}$$

Ex. Upwind method, $a > 0$.

$$v_j^{n+1} = v_j^n - C_a \left[v_j^n - v_{j-1}^n \right], \quad C_a = \frac{a \Delta t}{\Delta x}$$

$$\rightarrow e^{\alpha (n+1) \Delta t} e^{I \xi j \Delta x} = e^{\alpha n \Delta t} e^{I \xi j \Delta x} - C_a e^{\alpha n \Delta t} e^{I \xi j \Delta x} \left[1 - e^{-I \xi \Delta x} \right],$$

$$\rightarrow e^{\alpha \Delta t} = 1 - C_a \left[1 - e^{-I \xi \Delta x} \right]$$

$$= 1 - C_a \left[1 - \cos(\xi \Delta x) - I \sin(\xi \Delta x) \right]$$

$$= \left(1 - C_a + C_a \cos(\xi \Delta x) \right) - I \, C_a \sin(\xi \Delta x)$$

$\rightarrow$ Want : $|e^{\alpha \Delta t}| \leq 1$

$$\rightarrow 1 \geq |e^{\alpha \Delta t}|^2 = (1 - C_a + C_a \cos(\xi \Delta x))^2 + (C_a \sin(\xi \Delta x))^2$$

$$= 1 + C_a^2 + C_a^2 \cos^2(\xi \Delta x) - 2C_a + 2C_a(1 - C_a)\cos(\xi \Delta x) + C_a^2 \sin^2(\xi \Delta x)$$

$$= 1 + 2C_a^2 - 2C_a - 2C_a(1 - C_a)\cos(\xi \Delta x)$$

$$= 1 - 2C_a(1 - C_a) - 2C_a(1 - C_a)\cos(\xi \Delta x)$$

$$= 1 - 2C_a(1 - C_a)\underbrace{(1 - \cos(\xi \Delta x))}_{\geq 0}$$

$$\rightarrow 0 \geq -2C_a(1 - C_a)\underbrace{(1 - \cos(\xi \Delta x))}_{\geq 0}$$

$$\rightarrow 0 \leq C_a(1 - C_a)$$

$$\rightarrow \boxed{0 \leq C_a \leq 1}.$$

(!) upwind method is stable if $0 \leq \frac{a \Delta t}{\Delta x} \leq 1$.

Ex. Lax-Friedrichs method

$$U_j^{n+1} = \frac{1}{2}\left[U_{j+1}^n + U_{j-1}^n\right] - \frac{Ca}{2}\left[U_{j+1}^n - U_{j-1}^n\right]$$

→ Going through the same procedure and dividing by $e^{\alpha n\Delta t}e^{I\xi j\Delta x}$:

$$e^{\alpha\Delta t} = \frac{1}{2}\left[e^{I\xi\Delta x} + e^{-I\xi\Delta x}\right] - \frac{Ca}{2}\left[e^{I\xi\Delta x} - e^{-I\xi\Delta x}\right]$$

$$= \cos(\xi\Delta x) - Ca\, I\, \sin(\xi\Delta x)$$

$$\rightarrow 1 \geq \left|e^{\alpha\Delta t}\right|^2 = \cos^2(\xi\Delta x) + Ca^2\sin^2(\xi\Delta x)$$

$$= 1 - \sin^2(\xi\Delta x) + Ca^2\sin^2(\xi\Delta x)$$

$$= 1 + (Ca^2 - 1)\sin^2(\xi\Delta x), \quad \text{since } 0 \leq \sin^2(\xi\Delta x) \leq 1$$

$$\rightarrow 0 \geq (Ca^2 - 1)\sin^2(\xi\Delta x)$$

$$\rightarrow (Ca^2 - 1) \leq 0$$

$$\rightarrow (Ca - 1)(Ca + 1) \leq 0$$

$$\rightarrow \boxed{-1 \leq Ca \leq 1}$$

→ LF is stable for $-1 \leq \frac{a\Delta t}{\Delta x} \leq 1$.

Ex. Lax-Wendroff method.

$\rightarrow$ LW is stable for $-1 \leq \frac{a\Delta t}{\Delta x} \leq 1$. (HW)

Ex. Leapfrog method

$\rightarrow$ Leapfrog is stable for $-1 < \frac{a\Delta t}{\Delta x} < 1$. (HW)

[4] Local truncation error E_{LT}^{n+1} & orders of accuracy

Ex, Upwind method for $a>0$,

$$U_i^{n+1} = U_i^n - C_a \left[U_i^n - U_{i-1}^n \right], \quad C_a = \frac{a\Delta t}{\Delta x},$$

$$\rightarrow E_{LT}^{n+1} = \frac{1}{\Delta t}\left[u(x_i, t^{n+1}) - u(x_i, t^n) \right] + \frac{a}{\Delta x}\left[u(x_i, t^n) - u(x_{i-1}, t^n) \right]$$

$$= \frac{1}{\Delta t}\left[\cancel{u(x_i,t^n)} + \Delta t\, u_t(x_i,t^n) + \frac{\Delta t^2}{2} u_{tt}(x_i,t^n) + O(\Delta t^3) - \cancel{u(x_i,t^n)} \right]$$

$$+ \frac{a}{\Delta x}\left[\cancel{u(x_i,t^n)} - \left\{ \cancel{u(x_i,t^n)} - \Delta x\, u_x(x_i,t^n) + \frac{\Delta x^2}{2} u_{xx}(x_i,t^n) + O(\Delta x^3) \right\} \right]$$

$$= u_t(x_i,t^n) + \frac{\Delta t}{2} u_{tt}(x_i,t^n) + O(\Delta t^2)$$

① ($u_t + a u_x = 0$)

$$+ a\, u_x(x_i,t^n) - a\Delta x\, u_{xx}(x_i,t^n) + O(\Delta x^2)$$

$$\underbrace{u_t(x_i,t^n) + a\, u_x(x_i,t^n)}_{=\,0}$$

② $u_t = -a u_x$

$u_{tt} = -a u_{tx}$

$= -a(-a u_x)_x$

$= a^2 u_{xx}$

$$= \frac{a^2 \Delta t}{2} u_{xx}(x_i,t^n) - a\Delta x\, u_{xx}(x_i,t^n) + O(\Delta t^2 + \Delta x^2)$$

$$= \left(\frac{a^2 \Delta t}{2} - a\Delta x \right) u_{xx}(x_i,t^n) + O(\Delta t^2 + \Delta x^2)$$

$$\rightarrow E_{LT}^{n+1} = O(\Delta t + \Delta x)$$

→ Upwind method is <u>1st order in both space & time</u>.

→ Also, $\lim_{\Delta t, \Delta x \to 0} E_{LT}^{n+1} = 0$ (i.e, consistent) as long as

u is twice differentiable.

Ex. Show that the Lax-Friedrichs method is 1^{st}-order in time & 2^{nd} order in space.
(This means that LF is only 1^{st}-order in solving the advection eqn.)

Also, show that the method is consistent. (HW)

Ex. Show that the Lax-Wendroff method is 2^{nd} order in solving the advection eqn (i.e., 2^{nd} order in both space & time).

Also, show that the method is consistent. (HW)