Finite Difference methods for linear scalar advection.

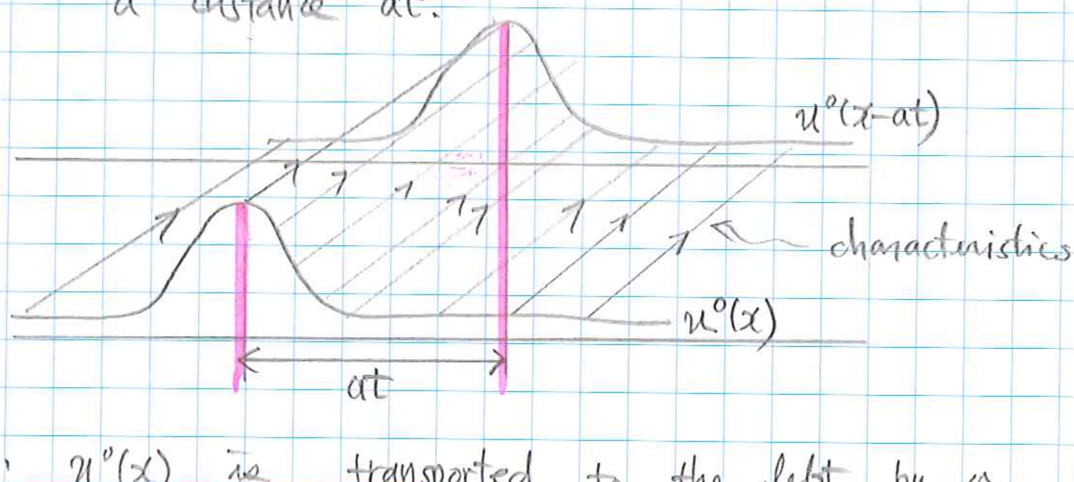
→ We are interested in numerically solving a linear scalar advection given by, on $[x_a, x_b]$,

$$\boxed{u_t + a u_x = 0}, \quad a > 0 \text{ or } a < 0,$$

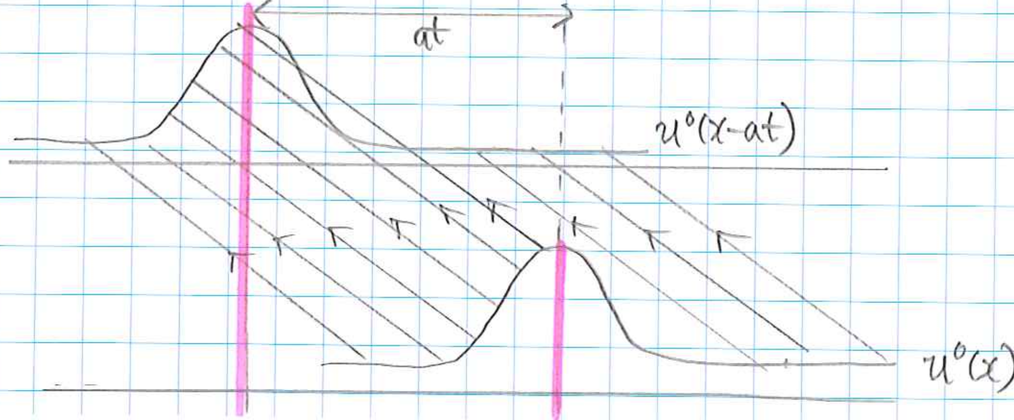
$$\begin{cases} \text{IC}: & u(x,0) = u^0(x) \\ \text{BC}: & \begin{cases} u(x_a,t) = g_a(t), \\ u(x_b,t) = g_b(t) \end{cases} \end{cases}$$

→ Since the exact soln is $u(x,t) = u^0(x-at)$,

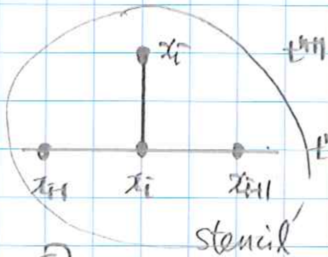
(i) if $a>0$; $u^0(x)$ is transported to the right by a distance "at".



(ii) if $a<0$; $u^0(x)$ is transported to the left by a distance "at".



Ex. If we use FTCS for Eqn. ① :



$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = -\frac{a}{2\Delta x}\left[U_{i+1}^n - U_{i-1}^n\right], \quad \text{or}$$

$$\boxed{U_i^{n+1} = U_i^n - \frac{a\Delta t}{2\Delta x}\left[U_{i+1}^n - U_{i-1}^n\right]} \quad \cdots ⑤$$

→ The method is $O(\Delta t + \Delta x^2)$ (HW), similar to the FTCS for the heat eqn.

→ Contrary to the case with the heat eqn, FTCS for the advection eqn is NOT stable, which will be examined soon later. → two-level method

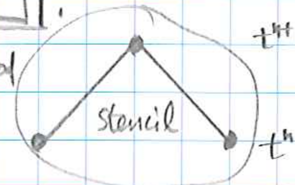
Ex. A minor modification of ⑤ is available to produce a bit more improved method, called "Lax-Friedrichs" (LF) method.

→ The modification is to replace, in ⑤,

$$U_i^n \quad \text{with} \quad \frac{U_{i+1}^n + U_{i-1}^n}{2}, \quad \text{giving}$$

$$\boxed{U_i^{n+1} = \frac{1}{2}\left[U_{i+1}^n + U_{i-1}^n\right] - \frac{a\Delta t}{2\Delta x}\left[U_{i+1}^n - U_{i-1}^n\right]}.$$

→ two-level method



→ LF is $O(\Delta t + \Delta x^2)$. (HW)

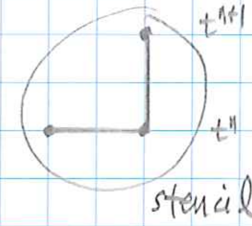
→ Although improved, LF is NOT practically useful. The major importance of LF is to provide a guidance on some stability issues which will be examined later as well.

→ LF is stable & convergent if $\left|\frac{a\Delta t}{\Delta x}\right| \le 1$. (shown later).

Ex FTBS (Forward in time, Backward in space) ~ $O(\Delta t + \Delta x)$

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + a\left[\frac{v_i^n - v_{i-1}^n}{\Delta x}\right] = 0, \quad \text{or}$$

$$\boxed{v_i^{n+1} = v_i^n - \frac{a\Delta t}{\Delta x}\left[v_i^n - v_{i-1}^n\right] = 0}$$

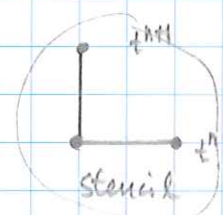


→ two-level method

Ex. FTFS (Forward in time, Forward in space) ~ $O(\Delta t + \Delta x)$

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + a\left[\frac{v_{i+1}^n - v_i^n}{\Delta x}\right] = 0, \quad \text{or}$$

$$\boxed{v_i^{n+1} = v_i^n - \frac{a\Delta t}{\Delta x}\left[v_{i+1}^n - v_i^n\right] = 0}$$

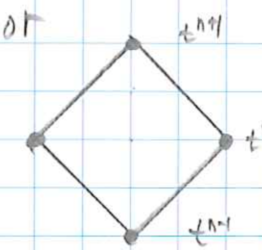


→ two-level method

Ex. Leapfrog ~ $O(\Delta t^2 + \Delta x^2)$

$$\frac{v_i^{n+1} - v_i^{n-1}}{2\Delta t} + a\left[\frac{v_{i+1}^n - v_{i-1}^n}{2\Delta x}\right] = 0, \quad \text{or}$$

$$\boxed{v_i^{n+1} = v_{i-1}^{n-1} - \frac{a\Delta t}{\Delta x}\left[v_{i+1}^n - v_{i-1}^n\right] = 0}$$



→ three-level method

Ex. Lax-Wendroff ~ $O(\Delta t^2 + \Delta x^2)$

→ Consider a given the semidiscrete form with CS ;

$$\frac{du(x_i,t)}{dt} = -a\left[\frac{u(x_{i+1},t) - u(x_{i-1},t)}{2\Delta x}\right]$$

which is exact in time, & $O(\Delta x^2)$ in space,

→ To achieve 2^{nd} order accuray on the advection eqn (meaning 2^{nd} order in both time & space), we additionally need 2^{nd} order temporal discretization.

→ The Leapfrog method achieves this by using the midpoint rule for time, giving three-level method.

→ Can we obtain $O(\Delta t^2 + \Delta x^2)$ in two-level method?

→ We again consider the idea of Taylor series methods in ODE:

$$\underbrace{u(t^{n+1}) = u(t^n) + \Delta t\, u'(t^n) + \frac{\Delta t^2}{2} u''(t^n)}_{(5)} + O(\Delta t^3)$$

⇒ this provides the 2^{nd} order method in time.

→ Recall that MOL (method of lines), applied to the semi discrete form of the advection eqn, yields

$$\boxed{\frac{d\underset{\sim}{U}(t)}{dt} = \underset{\sim}{A}\,\underset{\sim}{U}(t) + \underset{\sim}{g}(t)} \quad (6)$$

where

$$\underset{\sim}{A} = -\frac{a}{2\Delta x}\begin{bmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{bmatrix}_{N\times N}, \quad \underset{\sim}{U}(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_N(t) \end{bmatrix}_{N\times 1}$$

$$\underset{\sim}{g}(t) = -\frac{a}{2\Delta x}\begin{bmatrix} -U_0(t) \\ 0 \\ \vdots \\ 0 \\ U_{N+1}(t) \end{bmatrix}_{N\times 1} = \frac{a}{2\Delta x}\begin{bmatrix} g_a(t) \\ 0 \\ \vdots \\ 0 \\ g_b(t) \end{bmatrix} \leftarrow \text{BC}$$

→ To simplify further discussion, let's suppose we're given a periodic BC :

$$\Rightarrow \begin{cases} U_0(t) = U_{N-1}(t) \\ U_{N+1}(t) = U_1(t) \end{cases}$$

→ Then we can include $\underset{\sim}{g}(t)$ into $\underset{\sim}{\hat{A}}$, giving a simpler form

$$\boxed{\frac{d\underset{\sim}{U}}{dt} = \underset{\sim}{\hat{A}}\, \underset{\sim}{U}(t)} \quad \cdots ⑦ , \text{ where}$$

BC

$$\underset{\sim}{\hat{A}} = -\frac{a}{2\Delta x}\begin{bmatrix} 0 & 1 & & & & -1 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & & \ddots & & \\ & & & -1 & 0 & 1 \\ 1 & & & & -1 & 0 \end{bmatrix}$$

BC

→ From ⑦, $\underset{\sim}{U}'(t) = \hat{A}\, U(t) \Rightarrow \underset{\sim}{U}''(t) = \hat{A}\, \underset{\sim}{U}'(t) = \hat{A}^2 \underset{\sim}{U}(t)$.

→ Using this in the Taylor series expansion ⑤ up to $O(\Delta t^2)$, we get

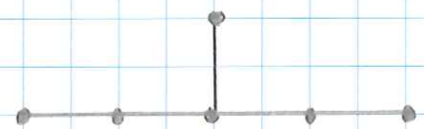
$$\boxed{\underset{\sim}{U}(t^{n+1}) = \underset{\sim}{U}(t^n) + \Delta t\, \underset{\sim}{\hat{A}}\, U(t^n) + \frac{\Delta t^2}{2} \underset{\sim}{\hat{A}}^2\, U(t^n)} \quad \cdots ⑧$$

→ ⑧ is the 2nd order method in both time & space. (Note the 2nd order in space is due to the 2nd order spatial discretization reflected in $\underset{\sim}{\hat{A}}$).

→ Computing $\hat{A}^2$ and writing the method at grid pts ;

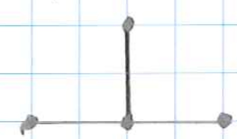
$$U_i^{n+1} = U_i^n + \Delta t\left(\frac{a}{-2\Delta x}\right)\left[U_{i+1}^n - U_{i-1}^n\right] + \frac{\Delta t^2}{2}\left(\frac{a}{-2\Delta x}\right)^2\left[U_{i-2}^n - 2U_i^n + U_{i+2}^n\right]$$

→ Note that this method needs a large 5-pt stencil ;



due to the last term which approximates $\frac{a^2}{2}\Delta t^2 u_{xx}$ using a centered differencing with $2\Delta x$,

→ Instead, we can approximate it on a more compact 3-pt stencil ;



obtaining the Lax-Wendroff method (LW)

$$\boxed{U_i^{n+1} = U_i^n - \frac{a\Delta t}{2\Delta x}\left[U_{i+1}^n - U_{i-1}^n\right] + \frac{\Delta t^2}{2}\left(\frac{a}{-\Delta x}\right)^2\left[U_{i-1}^n - 2U_i^n + U_{i+1}^n\right]}$$

Rmk. One can derive the LW method in a simpler way ;

Given $u_t + a u_x = 0$,

→ Apply Taylor series expansion directly on the PDE ;

$$u(x, t+\Delta t) = u(x,t) + \Delta t\, u_t(x,t) + \frac{\Delta t^2}{2} u_{tt}(x,t) + O(\Delta t^3) \quad \cdots (9)$$

→ Use $\begin{cases} u_t = -a u_x, \quad \& \\ u_{tt} = (-a u_x)_t = (-a)(u_t)_x = (-a)(-a u_x)_x = a^2 u_{xx} \end{cases}$

→ (9) becomes $\cdots$ (10)

$$u(x, t+\Delta t) = u(x,t) - a\Delta t\, u_x(x,t) + \frac{\Delta t^2}{2} a^2 u_{xx}(x,t) + O(\Delta t^3)$$

→ LW is then easily achieved by using standard centered differencing approximations to u_x & u_{xx}:

$$\begin{cases} u_x \cong \frac{1}{\Delta x}\left[u(x+\Delta x,t) - u(x-\Delta x,t) \right], \\ u_{xx} \cong \frac{1}{\Delta x^2}\left[u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t) \right] \end{cases}$$

and dropping terms with $\mathcal{O}(\Delta t^3)$ and higher.

$\blacksquare$

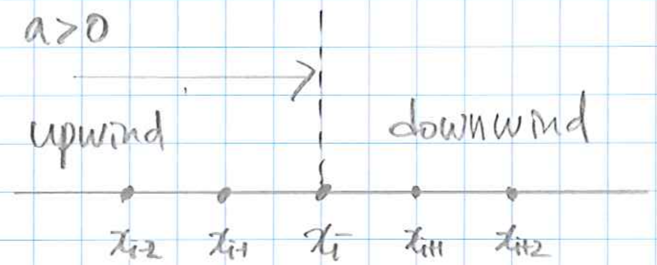
Ex. Upwind method $\sim O(\Delta t + \Delta x)$

→ First we define the upwind direction.

Def. The upwind direction is the direction facing the wind. The opposite direction is called the downwind direction.

→ We can thefore have different discrete formulations depending on the wind direction.

→ Case 1: $a > 0$



⇒ At $x = x_i$, the wind has already swept the upwind region $\{x \mid x \le x_i\}$.

⇒ the first-order upwind method is to use the data in the upwind region only for the $a u_x$ approximation.

current location

upwind information

$$\Rightarrow \frac{U_i^{n+1} - U_i^n}{\Delta t} + a \frac{U_i^n - U_{i-1}^n}{\Delta x} = 0, \quad \text{or}$$

upwind: $$\boxed{U^{n+1} = U_i^n - \frac{a\Delta t}{\Delta x}\left[U_i^n - U_{i-1}^n\right] = 0} \quad \cdots (11)$$

⇒ On the other hand, the downwind method becomes

downwind information

current information

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + a \frac{U_{i+1}^n - U_i^n}{\Delta x} = 0, \quad \text{or}$$

downwind: $$\boxed{U_i^{n+1} = U_i^n - \frac{a\Delta t}{\Delta x}\left[U_{i+1}^n - U_i^n\right] = 0} \quad \cdots (12)$$

→ Case 2: $a < 0$: the rolls are changed and hence

- upwind: (12)
- downwind: (11)

→ The crucial importance on the upwind method is that the method provides the proper stability depending on the direction of advection.

→ This stability gain in the upwind method is much more important than anything, although it seems, at first glance, to be less accurate than the standard centered spatial differencing.

→ We will see, using Von Neumann stability analysis, that

- upwind → stable
- downwind → unstable
- centered in space → unstable

→ In summary, for the advection eqn, a difference method that includes information in the downwind region, the method becomes unstable.