# §3. Hypberbolic PDEs

→ Hyperbolic PDEs arise in many physical problems whenever "Wave" motion is present;

(i) acoustic waves,
(ii) electromagnetic waves,
(iii) seismic waves,
(iv) shock waves, etc.

→ Real application problems are mostly nonlinear hyperbolic PDEs, but in practice, we "linearize" them at local grid points $(x_i, t^n)$ and solve the resulting linearized hyperbolic PDEs.

→ In this chapter, we are interested in the simplest possible case, a linear scalar first order hyperbolic PDE, in particular, "a linear scalar advection eqn."

→ We will learn "a system of (non)linear hyperbolic PDEs" in AMS 260.

## II Advection

refers to the fact that "a" = constant.

- Consider the scalar advection eqn:

$$\boxed{u_t + a u_x = 0} \quad \text{--- ①}$$

, $a$ : a signed constant. (advection velocity),

with (IC) $\boxed{u(x,0) = u^0(x)}$ ,

- For the Cauchy problem, we do not need any BC, but otherwise, we also need BCs:

(BC) $\boxed{\begin{aligned} u(x_a,t) &= g_a(t) \\ u(x_b,t) &= g_b(t) \end{aligned}}$ for 1D on $[x_a, x_b]$, $t>0$

$g_a(t)$ & $g_b(t)$ : ftns for BCs at $x = x_a$ & $x = x_b$, respectively.

- Note that the exact soln of ① is given as

$$\boxed{u(x,t) = u^0(x-at)} \quad , \ t \geq 0.$$

~~proof~~. let $\xi = x - at$.

$\rightarrow \dfrac{\partial \xi}{\partial t} = -a, \quad \dfrac{\partial \xi}{\partial x} = 1$

$$\rightarrow \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \frac{du^0}{d\xi}\frac{\partial \xi}{\partial t} + a\frac{du^0}{d\xi}\frac{\partial \xi}{\partial x}$$
$$= \frac{du^0}{d\xi}(-a) + a\frac{du^0}{d\xi}(1)$$
$$= 0.$$

$\therefore$ $u^0(x-at)$ is the exact soln of ①. ▨

Def. $\boxed{x - at = x_0}$ is called a <u>characteristic line</u> with a given constant $x_0$ and with the advection velocity $a$.

Rmk. Depending on the sign of $a$, the initial data $u^0(x)$ is advected;

(i) to the right if $a > 0$,

(ii) to the left if $a < 0$.

Rmk. Note that there are $\infty$ many characteristic lines in the $x$-$t$ plane as there are $\infty$ many choices of $x_0 \in \mathbb{R}$.

Rmk. Important property of the char. line:

→ the sln $u(x,t)$ remains as <u>constant over time along the char. lines.</u>

proof.

$$
\begin{aligned}
\frac{d}{dt} u(x,t) &= \frac{d}{dt} u(x(t),t) \\
&= \frac{\partial}{\partial t} u(x(t),t) \underbrace{\frac{dt}{dt}}_{=1} + \frac{\partial}{\partial x} u(x(t),t) \underbrace{\frac{\partial x}{\partial t}}_{=a} \\
&= u_t + a u_x \\
&= 0.
\end{aligned}
$$

$\therefore$ $u(x,t)$ is constant along char. lines, over time.

Rmk. If we consider a more complicated problem

$$\boxed{a = a(x) \neq \text{constant}},$$

this is NOT a scalar avection, and the soln constancy along the characteristics is no longer to be true.

proof. → We now have $\boxed{u_t + (a(x)u)_x = 0.}$ ⋯ ②

→ The characteristics are no longer to be straight lines but curves, since the advection velocity $a(x)$ changes in $x$.

→ The char curves $x(t)$ is a soln to ODE:

$$\begin{cases} \dfrac{dx(t)}{dt} = a(x(t)) & \cdots ③ \\ x(0) = x_0 \end{cases}$$

→ Also, the soln $u(x,t)$ is no longer to be constant along the chars $x(t)$:

~~proof~~.

Note ② $\Leftrightarrow$ $0 = u_t + a(x)u_x + a'(x)u$ ⋯ ④

$$\Rightarrow \frac{d}{dt}u(x,t) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}$$

$$= u_t + u_x \cdot a(x) \quad ③$$

$$= -a'(x)u \quad ④$$

$$\neq 0.$$