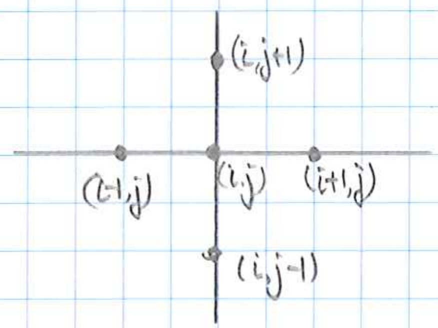
5. Multi-dimensional problem (2D)

→ $u_t = k(u_{xx} + u_{yy})$ on a bounded domain Ω,

$$\begin{cases} IC;\ u(x,y,0) = u^0(x,y) \\ BC;\ u(x,y,t) = g(x,y), \quad \forall x,y \in \partial\Omega. \end{cases}$$

→ Introducing the difference operators in x & y;

$$\begin{cases} D_x^2 U_{i,j} = \frac{1}{\Delta x^2}\left[U_{i-1,j} - 2U_{i,j} + U_{i+1,j} \right] \\ D_y^2 U_{i,j} = \frac{1}{\Delta y^2}\left[U_{i,j-1} - 2U_{i,j} + U_{i,j+1} \right] \end{cases}$$



; 5-pt stencil

→ If assuming $\Delta x = \Delta y = h$, we get a semi-discrete form;

$$\frac{dU_{i,j}}{dt} = k\left(D_x^2 + D_y^2\right) U_{i,j} \quad \text{--- ①}$$

Ex If using Forward Euler in time for ①:

$$U_{i,j}^{n+1} = U_{i,j}^{n} + \frac{\Delta t}{h^2}\underbrace{\left[\begin{array}{ccccc} & & U_{i,j+1}^{n} & & \\ & & + & & \\ U_{i-1,j}^{n} & - & 4U_{i,j}^{n} & + & U_{i+1,j}^{n} \\ & & + & & \\ & & U_{i,j-1}^{n} & & \end{array}\right]}_{=(D_x^2+D_y^2)U_{i,j}^{n}}$$

Ex If using Trapezoidal method in time for ①:

$$U_{i,j}^{n+1} = U_{i,j}^{n} + \frac{\Delta t}{2}\left[D_x^2\left(U_{i,j}^{n} + U_{i,j}^{n+1}\right) + D_y^2\left(U_{i,j}^{n} + U_{i,j}^{n+1}\right)\right]$$

$$= U_{i,j}^{n} + \frac{\Delta t}{2}\left[\left(D_x^2 + D_y^2\right)U_{i,j}^{n} + \left(D_x^2 + D_y^2\right)U_{i,j}^{n+1}\right].$$

$\Rightarrow$ Crank-Nicolson in 2D.

Rmk. Stability limit in 2D

In 1D : $\dfrac{K\Delta t}{\Delta x^2} \le \dfrac{1}{2}$, or $\Delta t \le \dfrac{\Delta x^2}{2K}$

$\Rightarrow$ In 2D : $K\Delta t\left(\dfrac{1}{\Delta x^2} + \dfrac{1}{\Delta y^2}\right) \le \dfrac{1}{2}$, or $\Delta t \le \dfrac{1}{2K}\left(\dfrac{1}{\Delta x^2} + \dfrac{1}{\Delta y^2}\right)^{-1}$

So, if $\Delta x = \Delta y = h$; $\quad \Delta t \le \dfrac{1}{2K}\left(\dfrac{2}{h^2}\right)^{-1} = \dfrac{h^2}{4K}$.

Ex. One can also use explicit & implicit methods separately for x & y directions

⇒ Alternating Direction Implicit (ADI) :

$$\begin{cases} U_{ij}^{n+\frac{1}{2}} = U_{ij}^{n} + \frac{\Delta t}{2}\left(D_x^2 U_{ij}^{n+\frac{1}{2}} + D_y^2 U_{ij}^{n}\right) & \cdots ② \\ U_{ij}^{n+1} = U_{ij}^{n} + \frac{\Delta t}{2}\left(D_x^2 U_{ij}^{n+\frac{1}{2}} + D_y^2 U_{ij}^{n+1}\right) & \cdots ③ \end{cases}$$

⇒ ② : implicit advancement by $\Delta t/2$ in x-direction with explicit in y

③ : Finalize with implicit advancement by $\Delta t/2$ in y with explicit in x.

⇒ ② : first-order in $\Delta t/2$ in x-direction

③ : another first-order in $\Delta t/2$ in y-direction

⇒ With the symmetry of the two-steps, the local truncation error introduced in ③ almost exactly cancels out the error in ①.

⇒ 2^{nd} order in Δt.