4 Von Neumann Stability Analysis

→ The von Neumann analysis is based on Fourier analysis and hence it is generally limited to <u>constant coeff. linear PDEs.</u>

→ It usually is applied to the <u>Cauchy Problem</u>, which is the PDE on all space with no boundaries,

$-\infty < x < \infty$ in 1D cases.

→ The Cauchy problem for linear PDEs can be solved using Fourier transforms (FT):

⇒

$$
\begin{tikzcd}
\underline{\text{Real space}} & & \underline{\text{Fourier space}} \\
\text{PDE} \arrow[rr, "\text{FT}"] & & \text{ODE} \arrow[d] \\
\text{Soln to the PDE} & & \text{find the soln to ODE} \arrow[ll, "\text{Inverse FT}"']
\end{tikzcd}
$$

⇒ the mathematical reason how we can use FT to solve the Cauchy Problem is because the fns

$$e^{I\xi x} = \sin(\xi x) + I\cos(\xi x),$$

$$\begin{cases} I = \sqrt{-1}, \\ \xi = \text{wave \#, constant} \end{cases}$$

are <u>eigenftns</u> of the differential operator ∂_x:

$$\partial_x \left(e^{I\xi x}\right) = (I\xi)\left(e^{I\xi x}\right) \quad \text{--- (1)}$$

Rmk. An eigenfun of a linear operator A defined on a fun space, is a non-zero fun f in that fun space satisfying

$$Af = \lambda f, \quad \text{for some scalar } \lambda.$$

→ With this FT property, we claim that if we let

$$U_j^n = A(t^n)\, e^{I\xi j\Delta x}, \quad \text{--- ②}$$

where $\underbrace{A(t^n)}_{\text{only a fun of } t.} = e^{\alpha n \Delta t}$, amplification factor,

then U_j^n is an eigen-fun of any translation-invariant finite different operator

→ For example, if we approximate $u'(x_j)$ by the 2nd order centered difference formula $D^0 U_j^n$;

$$D^0 U_j^n \overset{\text{def}}{\equiv} \frac{1}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right), \quad \text{--- ③}$$

assuming U_j^n is given by the single Fourier mode as in ②,

we obtain

$$D^0 U_j^n = \frac{A(t^n)}{2\Delta x}\left[e^{I\xi(j+1)\Delta x} - e^{I\xi(j-1)\Delta x}\right]$$

$$= \frac{1}{2\Delta x}\left[e^{I\xi\Delta x} - e^{-I\xi\Delta x}\right] A(t^n)\, e^{I\xi j\Delta x}$$

scalar eigenvalue

$$= \underbrace{\left(\frac{I}{\Delta x}\sin(\xi\Delta x)\right)}\, U_j^n \quad \text{eigenfun} \quad \text{--- ④}$$

→ That is to say, the expression ② which is based on a single Fourier mode ξ, allows that

<u>U_j^n becomes the eigenftn of the linear operator</u>

$D^0 \approx \frac{\partial}{\partial x}$.

→ However, in general, as noted in ③, if U_j^n is given different from ②, then

$$D^0 U_j^n \neq \lambda U_j^n, \quad \lambda : \text{scalar},$$

i.e, U_j^n is <u>NOT</u> an eigenftn of D^0.

→ Also note that in ④:

$$\frac{I}{\Delta x} \sin(\xi \Delta x) = \frac{I}{\Delta x}\left(\xi \Delta x - \frac{1}{3!}(\xi \Delta x)^3 + \mathcal{O}((\xi \Delta x)^5) \right)$$

$$= I\xi - \frac{I\xi^3}{6} \Delta x^2 + \cdots,$$

which agrees with the eigenvalue $I\xi$ of ∂_x in the first term.

→ let's now consider how we can apply von Neumann analysis to find the stability region.

→ We are going to use the following:

(i) We assume that our discrete soln is given by the single Fourier mode representation ② :

$$\boxed{U_j^n = A(t^n)\, e^{I \xi j \Delta x}} \quad \cdots ⑤$$

$A(t^n) = e^{\alpha n \Delta t}$

(ii) In ⑤, we see that the amplification factor is $e^{\alpha \Delta t}$. Therfore, we use

$$\boxed{\left| e^{\alpha \Delta t} \right| \leq 1} \quad \cdots ⑥$$

for the stability condition.

∵) stable if $\| U^{n+1} \| \leq \| U^n \|$

$$\rightarrow \left| \frac{e^{\alpha (n+1) \Delta t}\, e^{I \xi j \Delta x}}{e^{\alpha n \Delta t}\, e^{I \xi j \Delta x}} \right| \leq 1$$

$$\parallel$$

$$\left| e^{\alpha \Delta t} \right|.$$

Ex. Show FTCS is stable for solving $u_t = k u_{xx}$ if $\frac{k\Delta t}{\Delta x^2} \le \frac{1}{2}$.

proof. let $U_j^n = e^{\alpha n \Delta t} e^{I\xi j \Delta x}$ and consider

$$U_j^{n+1} = U_j^n + \frac{k\Delta t}{\Delta x^2}\left[U_{j-1}^n - 2U_j^n + U_{j+1}^n\right] \text{ for } u_t = k u_{xx}$$

$$\Rightarrow e^{\alpha(n+1)\Delta t} e^{I\xi j\Delta x} = e^{\alpha n\Delta t} e^{I\xi j \Delta x}$$

$$+ \frac{k\Delta t}{\Delta x^2}\left[e^{\alpha n \Delta t} e^{I\xi(j-1)\Delta x} - 2e^{\alpha n\Delta t} e^{I\xi j\Delta x} + e^{\alpha n \Delta t} e^{I\xi(j+1)\Delta x}\right]$$

$\Rightarrow$ Dividing by $e^{\alpha n\Delta t} e^{I\xi j\Delta x}$ both sides :

$$e^{\alpha\Delta t} = 1 + \frac{k\Delta t}{\Delta x^2}\left[e^{-I\xi\Delta x} - 2 + e^{I\xi\Delta x}\right], \quad C_a = \frac{k\Delta t}{\Delta x^2}$$

$$= 1 + C_a\left[2\cos(\xi\Delta x) - 2\right]$$

$\Rightarrow$ Since $-1 \le \cos(\xi\Delta x) \le 1$, $\forall\, \xi\Delta x$, we have

$$\rightarrow -4 \le 2\cos(\xi\Delta x) - 2 \le 0$$

$$\rightarrow -4C_a \le C_a\left[2\cos(\xi\Delta x) - 2\right] \le 0$$

$$\rightarrow 1 - 4C_a \le \underbrace{1 + C_a\left[2\cos(\xi\Delta x) - 2\right]}_{= e^{\alpha\Delta t}} \le 1$$

$$\rightarrow 1 - 4C_a \le e^{\alpha\Delta t} \le 1.$$

$\Rightarrow$ We see that $|e^{\alpha\Delta t}| \le 1$ if $\boxed{4C_a \le 2}$, or

$$\boxed{C_a \le \frac{1}{2}} \Leftrightarrow \boxed{\frac{k\Delta t}{\Delta x^2} \le \frac{1}{2}}$$