3 Stability theory, PDE vs. ODE

→ Recall from the MOL (method of line) approach, we convert a given PDE to an ODE.

→ From there, we can use the ODE stability theory to discuss the stability of the original PDE.

→ Recall that we had a system of ODEs!

$$\frac{d\underset{\sim}{U}(t)}{dt} = \underset{\sim}{A}\,\underset{\sim}{U}(t) + \underset{\sim}{g}(t), \quad \text{where} \quad \text{---} ①$$

$$\left[\begin{array}{l} \underset{\sim}{A} : N \times N \text{ matrix} \\ \underset{\sim}{U} : N \times 1 \text{ vector} \\ \underset{\sim}{g}(t) : N \times 1 \text{ vector} \end{array}\right.$$

→ We consider the eigenvalues of $\underset{\sim}{A}$, which turn out to be

$$\lambda_p = \frac{2K}{\Delta x^2}\left(\cos(p\pi\Delta x) - 1\right), \quad p = 1, 2, \cdots, N, \quad \text{---} ②$$

→ In contrast to the ODE theory,

[$\lambda_p = \lambda_p(\Delta x)$; a fn of Δx, &

the dimension of $\underset{\sim}{A}$ increases/decreases with the size of spatial resolution.

→ This property is different in a system of ODEs in which

(i) the size of the Jacobian matrix

$$\left(\frac{\partial f}{\partial u}\right)_{N\times N}, \quad u \in \mathbb{R}^N, \; f: \mathbb{R}^N \to \mathbb{R}^N$$

is fixed with N = # of dependent variables,

and

(ii) there is NO dependency on Δx of the eigenvalues λ_p of the Jacobian matrix.

(iii) We only needed zero-stability for convergence!

that is,

$$\lambda_p \Delta t \in \text{stability region } S, \quad p = 1, \dots, N$$

as $\Delta t \to 0$.

→ However, in PDEs, we have the stability factor (called the Courant-Fredrichs-Lewy condition, or CFL condition) is given as a combination of Δt & Δx:

(i) $k\dfrac{\Delta t}{\Delta x^2}$ for $u_t = k u_{xx}$ (heat eqn)

(ii) $a\dfrac{\Delta t}{\Delta x}$ for $u_t + a u_x = 0$ (advection eqn)

And it is NOT obvious if

$$k\frac{\Delta t}{\Delta x^2} \to 0 \quad \left(\text{or } a\frac{\Delta t}{\Delta x} \to 0\right) \quad \text{as } \Delta t \to 0.$$

[☺ consider when $\Delta t \approx \Delta x \approx h$, then

$$k \frac{\Delta t}{\Delta x^2} \approx k \frac{1}{h} \longrightarrow \infty \quad \text{as} \quad h \to 0.$$]

→ Therefore, in PDEs, we need a relation between Δt & Δx, in addition to $\Delta t \to 0$.

→ This is the reason that, in PDEs, the stability condition for convergence depends on the "shape" of the stability region (i.e., absolute stability in ODE sense), in which the PDE stability requirement seeks for the relation between Δt & Δx.

spatial information (i.e., Δx)

→ Returning to Eqn ②, since $-1 \le \cos(p\pi\Delta x) \le 1$, we see that

$$\lambda_p \in \{x \in \mathbb{R} \mid x \le 0\}, \quad \forall p = 1, \cdots, N$$

→ Also, the farthest eigenvalue from the origin is when $\cos(p\pi\Delta x) \approx -1$, giving

$$\lambda_p \approx \frac{-4k}{\Delta x^2}.$$

→ For stability, therefore, we require, for each p,

$$\boxed{z = \lambda_p \Delta t = \frac{-4k}{\Delta x^2}\Delta t \in S}, \quad S: \text{stability region of the ODE method.} \quad \cdots ③$$

Ex. If we use the Forward Euler's method for the semi-discrete form ①, then from the ODE theory, the stability region of FE ;

$$S = \{ z \in \mathbb{C} \mid |1+z| \le 1 \}, \quad \text{or}$$

$$S = \{ z \in \mathbb{R} \mid |1+z| \le 1 \}. \Leftrightarrow \boxed{-2 \le z \le 0}$$

Considering only the real case, we have from ③ ;

$$z = \lambda_p \Delta t \in S, \quad \text{for each } p$$

$$\Leftrightarrow \quad -2 \le \frac{-4k\Delta t}{\Delta x^2} \le 0$$

$$\Rightarrow \quad \boxed{\frac{k\Delta t}{\Delta x^2} \le \frac{1}{2}}$$