[4] Discretization

→ In PDEs, we need to discretize both

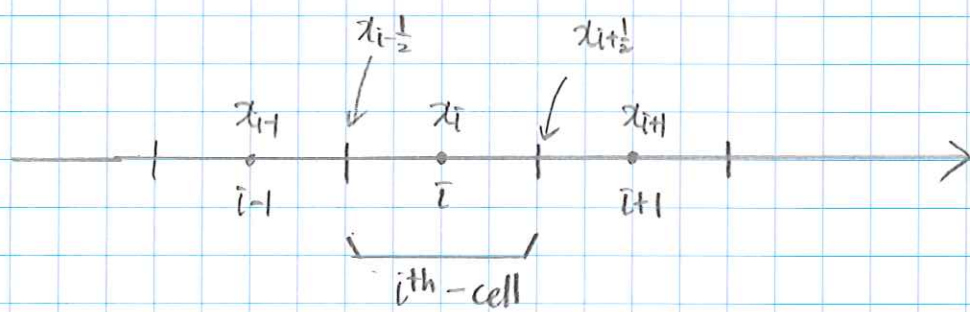
(i) time t, and
(ii) space, x, y, z

→ Notations are given as ;

(i) $t^n = n\Delta t$, $n = 0, \cdots, M$,
(ii) $x_i = (i - \frac{1}{2})\Delta x$, $i = 1, 2, \cdots, N$.

→ In addition, we also adopt the "half-integer" :

(iii) $x_{i+\frac{1}{2}} = x_i + \frac{\Delta x}{2}$ (x_i : cell-center
$x_{i+\frac{1}{2}}$: cell-interface



Def. Let $u_i^n = u(x_i, t^n)$ be the pointwise values of the exact soln u of a given PDE at discrete pts (x_i, t^n).

→ No error is associated with u_i^n.

Def. Let U_i^n be the numerical approx to the exact soln u_i^n at discrete points (x_i, t^n) ;

$$\begin{cases} U_i^n \approx u_i^n & \text{for finite difference methods,} \\ U_i^n \approx \dfrac{1}{\Delta x}\displaystyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x)\,dx & \text{for finite volume methods.} \end{cases}$$

Def. D_i^n : the exact soln of the associated difference eqn of the PDE, e.g., for FTBS (forward in time backward in space) for the advection PDE :

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad \cdots \quad ①$$

the difference eqn. using FTBS is

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + a\,\frac{U_i^n - U_{i-1}^n}{\Delta x} = 0, \quad \cdots \quad ②$$

then D_i^n satisfies ② exactly :

$$\frac{D_i^{n+1} - D_i^n}{\Delta t} + a\,\frac{D_i^n - D_{i-1}^n}{\Delta x} = 0.$$

$\rightarrow$ D_i^n is the exact soln of the difference eqn with no round-off errors.

Def <u>Discretization error</u> at (x_i, t^n) :

$$E_{d,i}^n = u_i^n - D_i^n \qquad (= \text{truncation errors} + \text{BC errors})$$

Def. <u>Round-off error</u> at (x_i, t^n) :

$$E_{r,i}^n = D_i^n - U_i^n$$

(= repeated computer arithmetic operations with rounding off numbers to some significant digits)

Def <u>Global error</u> at (x_i, t^n) :

$$E_{g,i}^n = u_i^n - U_i^n$$

Rmk $E_{g,i}^n = E_{d,i}^n + E_{r,i}^n$.

Def The numerical method is <u>convergent</u> at t^n in $\|\cdot\|$ if

$$\lim_{\Delta t, \Delta x \to 0} \| E_g^n \| = 0$$

(ex) $$\| E_g^n \|_1 = \Delta x \sum_{i=1}^{N} | E_{g,i}^n | = \Delta x \sum_{i=1}^{N} | u_i^n - U_i^n |$$

(ex) $$\| E_g^n \|_p = \left(\Delta x \sum_{i=1}^{n} | E_{g,i}^n |^p \right)^{\frac{1}{p}}$$

[5] The Fundamental Theorem of Numerical Methods
— The Lax Equivalence Theorem for linear PDE.

<u>Thm</u>. (only for linear PDEs)

$$\underbrace{\text{Consistency} + \overset{\text{absolute}}{\text{stability}}}_{\text{easier to check without knowing the exact, analytical soln.}} \Leftrightarrow \underbrace{\text{convergence.}}_{\text{requires to know the exact soln.}}$$

<u>Def</u>. Let N be the <u>linear numerical operator</u> mapping the approximate soln at t^n to the approximate soln at t^{n+1}.

(e.g.) $U^{n+1} = N(U^n)$ → one step explicit.

<u>Def</u>. The <u>one-step error</u> $E^{n+1}_{1step,i}$ at (x_i, t^{n+1}),

$$E^{n+1}_{1step,i} = u^{n+1}_i - N(u^n_i), \quad i = 1, \cdots, N.$$

<u>Def</u>. The <u>local truncation error</u> $E^{n+1}_{LT,i}$ at (x_i, t^{n+1})

$$E^{n+1}_{LT,i} = \frac{1}{\Delta t} E^{n+1}_{1step,i}, \quad i = 1, \cdots, N.$$

Def. The numerical method is of order p in both t & x

for all sufficiently smooth data with compact support

$$\Leftrightarrow \quad E^{n+1}_{q,T} = \begin{cases} \mathcal{O}(\Delta t^p + \Delta x^p) & \text{in 1D} \\ \mathcal{O}(\Delta t^p + \Delta x^p + \Delta y^p) & \text{in 2D} \\ \mathcal{O}(\Delta t^p + \Delta x^p + \Delta y^p + \Delta z^p) & \text{in 3D.} \end{cases}$$

Rmk In practice one may have a method that is

$$\begin{cases} p^{th} \text{ order accurate in time, } \& \\ r^{th} \text{ order accurate in space} \end{cases}$$

$$\Rightarrow \quad E^{n+1}_{q,T} = \mathcal{O}(\Delta t^p + \Delta x^r)$$

→ In this case, the numerical soln in a fully resolved state, both temporally and spatially, will exhibite the convergence rate dominated by the larger error between the two:

$$\Rightarrow \quad E^{n+1}_{q,T} = \max\left[\mathcal{O}(\Delta t^p), \mathcal{O}(\Delta x^r)\right] \quad \cdots \quad ①$$

4/29/16

Def. The numerical method is consistent in $\|\cdot\|$ with ~~the target~~ PDE if

$$\lim_{\Delta t, \Delta x \to 0} \| E_{LT}^{n} \| = 0$$

for all smooth fns $u(x,t)$ that satisfies the given PDE.

Stability Theory for "Linear" PDEs

→ The stability Theory for nonlinear PDEs need yet different approches and we are not going to study them in AMS 213B. (But in AMS 260)

→ For linear PDEs, we need to bound the global error $E_{g,i}^{n+1} = u_i^{n+1} - U_i^{n+1}$ using a recurrence relation.

→
$$
\begin{aligned}
E_{g,i}^{n+1} &= u_i^{n+1} - U_i^{n+1} \\
&= u_i^{n+1} - N(U_i^{n}) \\
&= u_i^{n+1} - N(u_i^{n} - E_{g,i}^{n}) \\
&= \underline{u_i^{n+1} - N(u_i^{n})} + N(E_{g,i}^{n}) \\
&= E_{1step,i}^{n+1} + N(E_{g,i}^{n}) \\
&= \Delta t\, E_{LT,i}^{n+1} + N(E_{g,i}^{n}) \quad \cdots \quad ①
\end{aligned}
$$

Recall from the ODE theory, that we have:

<u>Def</u>. The linear numerical method defined by the the linear operator N is <u>stable</u> in $\|\cdot\|$ if

$\exists\, C$ s.t. $\quad \|N^n\| \leq C, \quad \forall n\,\Delta t \leq T,$

for each T.

<u>Rmk</u>. N^n is the <u>nth power</u> of the linear operator N obtained by the repeated applications of the linear operations of N.

→ this is only <u>true for linear operators</u>, but not for nonlinear operators.

<u>Rmk</u>. We also can say the numerical method is <u>stable</u> if $\|N\| \leq 1$, since

$$\|N^n\| \leq \|N\|^n \leq 1.$$

Thm. The Lax Equivalence Thm for linear difference methods

For a well-posed consistent linear method, (absolute) stability is necessary and sufficient condition for convergence:

$$\boxed{\text{Consistency} + \text{absolute stability} \iff \text{convergence}}$$

proof ($\Leftarrow$) Ritchtmyer and Morton,

Difference Methods for IVPs, Wiley-Interscience, 1967.

($\Rightarrow$) Claim: $\lim_{\Delta t, \Delta x \to 0} \| E_g^{n+1} \| = 0$.

[Note here that, if we use 1-norm,

$$\| E_g^{n+1} \| = \| E_g^{n+1} \|_1 = \Delta x \sum_{i=1}^{N} | E_{g,i}^{n+1} |$$
]

From Eq. ①:

$$\begin{aligned}
\| E_g^{n+1} \| &\le \Delta t \| E_{\tau}^{n+1} \| + \| N(E_g^n) \| \\
&\le \Delta t \| E_{\tau}^{n+1} \| + \| N \| \| E_g^n \| && \| AB \| \le \| A \| \| B \| \\
&\le \Delta t \| E_{\tau}^{n+1} \| + C \| E_g^n \| && \| N \| \le^{\exists} C \ (\because \text{stable}) \\
&\le \Delta t \| E_{\tau}^{n+1} \| + C \left(\Delta t \| E_{\tau}^n \| + C \| N \| \| E_g^{n-1} \| \right) \\
&\cdots \\
&\le \Delta t \sum_{j=1}^{n+1} C^{n+1-j} \| E_{\tau}^j \| + C^{n+1} \| E_g^0 \| \\
&\le \tilde{D} (n+1) \Delta t \| E_{\tau} \| + \tilde{C} \| E_g^0 \| \\
&= \tilde{D}\, t^{n+1} \| E_{\tau} \| + \tilde{C} \| E_g^0 \| \quad \cdots \text{②}
\end{aligned}$$

$\| E_{\tau} \| \overset{\text{let}}{\equiv} \max_j \| E_{\tau}^j \|$, for some constants $\tilde{C}$, $\tilde{D}$.

In Eqn ②,

(i) $\|E_g^0\| \to 0$ as $\Delta t, \Delta x \to 0$; otherwise it means that the problem is ill-posed due to the nonzero global error in resolving the initial data.

(ii) $\|E_T\| \to 0$ as $\Delta t \to 0$, since the method is consistent by assumption.

Therefore, we prove

$$\lim_{\Delta t, \Delta x \to 0} \|E_g^{n+1}\| = 0 \quad . \qquad \boxtimes$$

Rmk. We note that the above proof is only available when $\mathcal{N}$ is linear, hence the theorem is only valid for linear PDEs.

→ For nonlinear PDEs, we don't have any general form of stability theory, and we often just show nonlinear stability by showing that a suitable form of stability holds, using different techniques depending on the character of the PDEs.