Reviews on PDEs and Difference Equations.

[I] Properties of PDEs

- PDEs involve more than one "independent" variables, $t, x, y, z, \cdots$.

- Associated to them are called the "dependent" variables, say, u, which can be written as fns of the indep. vars:

$$u = u(t, x, y, z, \cdots)$$

Def. A PDE is a relation between the indep. vars and the dep. var(s) u via partial derivatives of u.

Def. The order of PDE is the highest derivative that appears.

Ex. The general form of first-order PDE in two indep. vars x & y:

$$F(x, y, u, u_x, u_y) = 0$$

Ex. The general form of second-order PDE in three indep. vars t, x, y:

$$F(t, x, y, u, u_t, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

Ex. $u_t - u_{xx} = 0$; parabolic PDE (or heat Eqn)

→ 2nd-order PDE in two indep. vars t & x.

Ex. $u_{xxxx} + (u_y)^3 = 0$; 4th-order PDE in two indep. vars x & y.

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Def. $\mathcal{L}$ is called a <u>linear operator</u> if

$$\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$$

for any fns u & v.

Def. A PDE $\mathcal{L}(u) = 0$ is called a <u>linear PDE</u> if $\mathcal{L}$ is a <u>linear derivative operator</u>.

Def. A PDE $\mathcal{L}(u) = g$ is called an <u>inhomogeneous linear PDE</u> if $\mathcal{L}$ is <u>a linear derivative operator</u> and if $g \neq 0$ is a <u>given fn of the indep. vars</u>.

If $g = 0$, then it is called a <u>homogeneous linear PDE</u>.

Ex. <u>Homogeneous linear PDEs</u>

(i) $u_t + u_x = 0$ (transport)

(ii) $u_t + x u_x = 0$ (transport)

(iii) $u_{xx} + u_{yy} = 0$ (Laplace's eqn).

Ex. Homogeneous nonlinear

nonlinear — interaction

(i) $u_t + uu_x = 0$ (Burgers' Eqn → shock wave)

(ii) $u_{tt} - u_{xx} + u^3 = 0$ (Wave with interaction)

(iii) $u_t + uu_x + u_{xxx} = 0$ (dispersive wave)

dispersive

Ex. Inhomog. linear

(i) $\cos(xy^2)\, u_x - y^2 u_y = \tan(x^2+y^2)$

[2] Well-posedness of PDEs

Def A PDE problem is said to be undetermined when too few conditions are imposed to the problem, resulting non-unique solns.

Def. A PDE problem is said to be overdetermined when two many conditions are given, resulting no soln at all (non-existence).

Def. Well-posed PDEs require proper initial and boundary conditions with the following properties:

(i) Existence : $\exists$ at least one soln $u(x,t)$,

(ii) Uniqueness : $\exists$ at most one soln $u(x,t)$,

(iii) Stability : the unique soln $u(x,t)$ depends in a stable manner on the data of the problem. This means that if the data are changed a little, the corresponding soln $u(x,t)$ changes a little as well.

[3] Classification of 2^{nd}-order PDEs

- PDEs arise in

  (i) fluid flows,

  (ii) heat transfer,

  (iii) solid mechanics,

  (iv) biological process, etc.

- PDEs fall into one of (or a combination of) the following three types :

  (i) hyperbolic PDE (e.g., $\underbrace{u_t + a u_x = 0}_{\text{linear}}$, $\underbrace{u_t + u u_x = 0}_{\text{nonlinear}}$) constant
  → (advection Eqn, wave Eqn)

  (ii) parabolic PDE (e.g., $u_t = k u_{xx}$)
  ⇒ (diffusion or heat Eqn).

  (iii) elliptic PDE (e.g., $u_{xx} + u_{yy} = h(x,y)$)
  → (Laplace's Eqn, Poisson Eqn)
  → steady state of (i) or (ii).

- In general, consider the PDE with nonzero constants a_{11}, a_{12} and a_{22} :

$$a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} + a_1 u_x + a_2 u_y + a_0 u = 0 \quad \cdots ①$$

Thm By a linear transformation of the indep. vars, the Eq. ① can be reduced to one of the three forms:

(i) Elliptic PDE : if $a_{12}^2 < a_{11} a_{22}$ it is reducible to

$$\boxed{u_{xx} + u_{yy} + L.O.T = 0}$$

all low order terms, first or zeroth order terms.

(ii) Parabolic PDE ; if $a_{12}^2 = a_{11} a_{22}$ it is reducible to

$$\boxed{u_{xx} + L.O.T = 0}$$

(iii) Hyperbolic PDE ; if $a_{12}^2 > a_{11} a_{22}$ it is reducible to

$$\boxed{u_{xx} - u_{yy} + L.O.T = 0}$$

Rmk. This classification is similar to the one in analytic geometry, given:

$$a_{11} x^2 + 2a_{12} xy + a_{22} y^2 + a_1 x + a_2 y + a_0 = 0 \quad \text{--- ②}$$

The Eqn ② is

(i) Ellipsoid if $a_{12}^2 < a_{11} a_{22}$ (e.g. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$)

(ii) Parabola if $a_{12}^2 = a_{11} a_{22}$ (e.g., $y = x^2$)

(iii) Hyperbola if $a_{12}^2 > a_{11} a_{22}$ (e.g., $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$)

Ex. (i) $u_{xx} - 5u_{xy} = 0$

$\rightarrow$ hyperbolic, since
$$D = a_{12}^2 - a_{11}a_{22} = \left(-\frac{5}{2}\right)^2 - (1)(0) = \frac{25}{4} > 0.$$

(ii) $4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0$

$\rightarrow$ parabolic, since
$$D = a_{12}^2 - a_{11}a_{22} = (-6)^2 - (4)(9) = 36 - 36 = 0.$$

(iii) $4u_{xx} + 6u_{xy} + 9u_{yy} = 0$

$\rightarrow$ elliptic, since
$$D = a_{12}^2 - a_{11}a_{22} = 3^2 - (4)(9) = 9 - 36 = -27 < 0.$$

Ex. The wave eqn is one of the most famous example in hyperbolic PDEs.

We write the wave eqn as:

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad c \neq 0 \quad \cdots (3)$$

$\rightarrow$ If we define the linear operator $\mathcal{L}$ by

$$\mathcal{L} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right), \quad \text{then}$$

$\rightarrow$ (3) is rewritten as

$$\mathcal{L}u = 0.$$

→ Considering so-called the characteristic coordinates

$$\begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases}$$

→ By the chain rule, we get

$$\Rightarrow u(x,t) \to u(\xi, \eta) = u(\xi(x,t), \eta(x,t))$$

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial x} = \dfrac{\partial u}{\partial \xi} + \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial t} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial t} = \dfrac{\partial u}{\partial \xi}(c) + \dfrac{\partial u}{\partial \eta}(-c) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial t} - c\dfrac{\partial u}{\partial x} = -2c\dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 2c\dfrac{\partial u}{\partial \xi} \end{cases}$$

$$\Rightarrow 0 = \mathcal{L}u = \left(-2c\frac{\partial}{\partial \eta}\right)\left(2c\frac{\partial}{\partial \xi}\right)u \quad \cdots \text{④}$$

⇒ Since $c \neq 0$, ④ means that

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \quad \cdots \text{⑤}$$

⇒ The soln of ⑤ is

$$\boxed{u = f(\xi) + g(\eta)} \quad \cdots \text{⑥}$$

for some f & g.

⇒ ⑥ is equivalent to

$$\boxed{u = f(x+ct) + g(x-ct)} \quad \cdots \text{⑦}$$

⇒ ⑦ implies that the general soln u of the wave eqn. ③ must have the form of the sum of the two ftns,

(i) g : a wave of any shape traveling to the right at speed c, &

(ii) f : a wave of any arbitrary shape traveling to the left at speed c.

⇒ We call the two families of lines,

$$\boxed{x \pm ct = \text{constant}} \quad \text{--- ⑧}$$

the characteristic lines of the wave eqn.

Ex. One very famous example in parabolic PDEs is so-called the diffusion eqn :

$$\boxed{u_t = k u_{xx}}, \quad k > 0, \text{ diffusion coefficient}$$

$$(x,t) \in D \times T, \quad \text{--- ⑨}$$

→ This eqn must satisfy the maximum principle :

if $u(x,t)$ ~~is~~ the soln of ⑨ on

$D \times T = [x_{min}, x_{max}] \times [T^0, T^1]$ in space-time,

then the maximum value of $u(x,t)$ only occurs either (i) initially at $t = T^0$, or

(ii) on the boundries at $x = x_{min}$ or x_{max}.

Property	Wave Eqn.	Diffusion Eqn
(1) speed of propagation	finite $\leq c$	∞
(2) singularities for $t > 0$?	transported along characteristics with speed c.	lost immediately
(3) well-posed for $t > 0$?	yes	yes (at least for bounded solns)
(4) well-posed for $t < 0$?	yes	No!
(5) maximum principle	No!	yes
(6) behavior as $t \to \infty$	energy is constant, and the initial wave profile does <u>NOT</u> decay in amplitude but keeps getting advected. $\to$ True theoretically, but not numerically.	decays to zero
(7) information	transported with characteristics	lost gradually