§1.10. Richardson's Extrapolation

- Richardson extrapolations are used for two cases;

  (i) predicting global errors

  $$E_g^n = u^n - U^n,$$

  without knowing the exact soln u^n, but only using U^n in an asymptotically correct sense,

  (ii) obtaining new high-order methods

  by recursively using a given low-order method to solve IVPs.

- We will learn both by considering an example. First, we need to state a theorem on the global error, due to Gragg.

Thm. Suppose that an explicit one-step method is of order p.

Assume $u(t)$ is the exact soln of the IVP

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(t^0) = u^0 \end{cases},$$

with all required smoothness properties of f to guarantee the soln existence & uniqueness.

Then the global error of the numerical method has an expansion in powers of Δt of the form

$$\boxed{E_g^n = u^n - U^n = c(t)(\Delta t)^p + O(\Delta t^{p+1})},$$

$c(t)$: a soln of $\quad c'(t) = f_u(t, u)c(t) + \delta u^{(p+1)}(t)$, δ: const

→ Let $U_{\Delta t}^n$ and $U_{2\Delta t}^n$ denote the numerical solns to the IVP on $[t^0, T]$, using step sizes Δt and $2\Delta t$, respectively, using the trapezoidal method,

$$U^{n+1} = U^n + \frac{\Delta t}{2}\left[f(U^n) + f(U^{n+1})\right].$$

→ Note that the trap method is 2nd-order.

→ According to the thm on the global error, we see that

$$u^n - U_{\Delta t}^n = G(t)\,\Delta t^2 + O(\Delta t^3) \quad \text{--- ①}$$

→ Using ①, we get

$$u^n - U_{2\Delta t}^n = 4G(t)\,\Delta t^2 + O(\Delta t^3) \quad \text{--- ②}$$

→ Multiplying ① by 4, subtract ②, and solve for u^n;

$$u^n = \frac{1}{3}\left[4U_{\Delta t}^n - U_{2\Delta t}^n\right] + O(\Delta t^3) \quad \text{--- ③}$$

→ Now if we let

$$\begin{cases} U^{n,(0)} = U^n & \leftarrow \text{num. soln of the trap method (2}^{\text{nd}}\text{-order)} \\ U_{\Delta t}^{n,(1)} \equiv \frac{1}{3}\left[4U_{\Delta t}^n - U_{2\Delta t}^n\right] & \\ \qquad\quad = \frac{1}{3}\left[4U_{\Delta t}^{n,(0)} - U_{2\Delta t}^{n,(0)}\right], & \text{--- ④} \end{cases}$$

We see that the newly defined soln $U^{n,(1)}$ in ④ is of order $\mathcal{O}(\Delta t^3)$, one order higher than the original soln $U^{n,(0)}$.

→ Rewriting ③ using ④, we simply get the global error of $U^{n,(1)}$, denoted as $E_g^{n,(1)}$, is given by

$$\begin{aligned} E_g^{n,(1)} &= u^n - U_{\Delta t}^{n,(1)} \\ &= C_2(t)\,\Delta t^3 + \mathcal{O}(\Delta t^4). \end{aligned}$$

→ We call $\{U_{j\Delta t}^{n,(1)}\}$ the Richardson extrapolate of $\{U_{j\Delta t}^{n,(0)}\}$, j = even numbers.

→ The sequence $U_{\Delta t}^{n,(1)}, U_{2\Delta t}^{n,(1)}, U_{4\Delta t}^{n,(1)}, \cdots$ is a new 3rd-order method.

→ Furthermore, if we consider $\{U_{2^k\Delta t}^{n,(k)}\}$, then we obtain so-called the Romberg rule, whose table is given as, (for $k = 0, 1, 2, \cdots$)

$$\begin{array}{llll}
U_{\Delta t}^{n,(0)} & & & \\
U_{2\Delta t}^{n,(0)} & U_{\Delta t}^{n,(1)} & & \\
U_{4\Delta t}^{n,(0)} & U_{2\Delta t}^{n,(1)} & U_{\Delta t}^{n,(2)} & \\
U_{8\Delta t}^{n,(0)} & U_{4\Delta t}^{n,(1)} & U_{2\Delta t}^{n,(2)} & \cdots \\
U_{16\Delta t}^{n,(0)} & U_{8\Delta t}^{n,(1)} & U_{4\Delta t}^{n,(2)} & \\
\vdots & \vdots & \vdots &
\end{array}$$

$$\rightarrow \begin{cases} U_{\Delta t}^{n,(1)} = \dfrac{4U_{\Delta t}^{n,(0)} - U_{2\Delta t}^{n,(0)}}{3}, \\ U_{2\Delta t}^{n,(1)} = \dfrac{4U_{2\Delta t}^{n,(0)} - U_{4\Delta t}^{n,(0)}}{3}, \\ \vdots \end{cases}$$

$$\begin{cases} U_{\Delta t}^{n,(2)} = \dfrac{16U_{\Delta t}^{n,(1)} - U_{2\Delta t}^{n,(1)}}{15} \\ U_{2\Delta t}^{n,(2)} = \dfrac{16U_{2\Delta t}^{n,(1)} - U_{4\Delta t}^{n,(1)}}{15} \\ \vdots \end{cases}$$

$$\vdots$$

And In general, we obtain

$$U_{2^k\Delta t}^{n,(k)} = \frac{4^k U_{2^k\Delta t}^{n,(k-1)} - U_{2^{k+1}\Delta t}^{n,(k-1)}}{4^k - 1}, \quad \text{and}$$

$$E_g^{n,(k)} = u^n - U^{n,(k)} = O(\Delta t^{2k+2})$$

→ The first column ; the Trap rule (2nd order)

the second column ; the Simpson rule (4th order),

etc.

→ Since each column converges to u^n more rapidly than the preceeding column, we see that $\{U^{n,(k)}_{2^k \Delta t}\}$, $k=0,1,2,\cdots$, converge more rapidly than all the previous values of k.

→ In many literature, this table is also called the <u>Romberg Integration</u> rule because the method is very popular to obtain high-order numerical integrations (or quadrature rules).

→ So far, we've looked at how to use the <u>Richardson extrapolation</u> to obtain <u>a new higher-order method</u>.

→ The other usage, as mentioned, is the <u>prediction of the global error</u>, which has been almost discussed so far, and let's complete the discussion here.

→ This can be done by realizing that the formula ③ is of greater use in predicting the global error $E_g^{n,(k)}$.

→ For instance, with $k=0$, we see that ③ gives

$$E_g^{n,(0)} = E_g^n$$
$$= u^n - U^n$$

note
$\left(U^n = U^n_{\Delta t} \right)$

$$= \frac{1}{3}\left[4U^n_{\Delta t} - U^n_{2\Delta t} \right] + O(\Delta t^3) - U^n$$
$$= \frac{1}{3}\left[U^n_{\Delta t} - U^n_{2\Delta t} \right] + O(\Delta t^3) \quad \cdots ⑤$$

→ The term $\frac{1}{3}\left[U^n_{\Delta t} - U^n_{2\Delta t} \right]$ which involves only two numerical solns $U^n_{\Delta t}$ & $U^n_{2\Delta t}$ provides an asymptotic estimate of the global error E_g^n for the trap method, without knowing anything about the exact soln.

Ex. Consider the IVP:

$$\begin{cases} u' = -u^2 \quad (= f(u)) \\ u(0) = 1. \end{cases}$$

→ The exact soln is $u(t) = \dfrac{1}{1+t}$.

→ The following table was generated using $\Delta t = \frac{1}{4}$, and $2\Delta t = \frac{1}{2}$.

→ The last column is the error estimate which is comparable to the true error E_g^n in the 5th column.

Table 6.8 Trapezoidal method and Richardson error estimation

x	$y_{2h}(x)$	$Y(x) - y_{2h}(x)$	$y_h(x)$	$Y(x) - y_h(x)$	$\frac{1}{3}[y_h(x) - y_{2h}(x)]$
1.0	.483144	.016856	.496021	.003979	.004292
2.0	.323610	.009723	.330991	.002342	.002460
3.0	.243890	.006110	.248521	.001479	.001543
4.0	.194838	.004162	.198991	.001009	.001051
5.0	.163658	.003008	.165937	.000730	.000759

↑ t

↑ $U_{2\Delta t}^n$

↑ $u^n - U_{2\Delta t}^n$

↑ $U_{\Delta t}^n$

↑ $u^n - U_{\Delta t}^n$, the true global error E_g^n.

↑ the global error estimate $\sim E_g^n$.

comparable!!!

Rmk. The Bulirsch–Stoer method (or sometimes called the Gragg–Bulirsch–Stoer method) is a method that combines the following three ideas

(i) Richardson extrapolation,

(ii) rational ftn extrapolation, and

(iii) the modified midpoint method.

→ Details are found in Chap. 16.3 & 16.4 in Numerical Recipes.