§1.9. Absolute stability

Ex. Consider the trapezoidal rules applied to a model problem

$$\begin{cases} u'(t) = \lambda u(t) \\ u(0) = 1 \end{cases}$$

The exact soln is $u(t) = e^{\lambda t}$.

We are interested in the following two cases:

(i) $\lambda \in \mathbb{R}$, $\lambda < 0$
(ii) $\lambda \in \mathbb{C}$, $Re(\lambda) < 0$
$\Big\}$ analytic solns asymptotically converge as $t \to \infty$.

and we are NOT interested in the following cases:

(iii) $\lambda \in \mathbb{R}$, $\lambda \geq 0$
(iv) $\lambda \in \mathbb{C}$, $Re(\lambda) \geq 0$
$\Big\}$ analytic solns do NOT asymptotically converge as $t \to \infty$.

Now consider the trapezoidal method applied to the IVP:

$$U^{n+1} = U^n + \frac{\Delta t}{2}\left[\lambda U^n + \lambda U^{n+1}\right]$$
$$= U^n + \frac{\lambda \Delta t}{2}\left[U^n + U^{n+1}\right]$$

$$\Rightarrow U^{n+1}\left(1 - \frac{\lambda \Delta t}{2}\right) = U^n\left(1 + \frac{\lambda \Delta t}{2}\right)$$

$$\Rightarrow U^{n+1} = \left(\frac{1 + \frac{\lambda \Delta t}{2}}{1 - \frac{\lambda \Delta t}{2}}\right) U^n$$

$$= \cdots$$

$$= \left(\frac{1 + \frac{\lambda \Delta t}{2}}{1 - \frac{\lambda \Delta t}{2}}\right)^n U^0$$

$$= \underbrace{\left(\frac{1 + \frac{\lambda \Delta t}{2}}{1 - \frac{\lambda \Delta t}{2}}\right)^n}_{= N^n} \quad \text{if } \frac{\lambda \Delta t}{2} \neq 0.$$

Let $\lambda = a + bi$ with $a < 0$, then

$$|r| = \left| \frac{1 + \frac{\lambda \Delta t}{2}}{1 - \frac{\lambda \Delta t}{2}} \right| = \left| \frac{2 + \lambda \Delta t}{2 - \lambda \Delta t} \right|$$

$$= \frac{\sqrt{(2 + a\Delta t)^2 + (b\Delta t)^2}}{\sqrt{(2 - a\Delta t)^2 + (b\Delta t)^2}}$$

< 1, since $(2 + a\Delta t)^2 < (2 - a\Delta t)^2$ when $a < 0$.

→ We call "$\underline{r}$" the amplification (or growth) factor.

→ Since $|r| < 1$, we see that $U^{n+1} \to 0$ as $n \to \infty$.

→ the method is <u>stable for all Δt</u>.

Rmk.
$$\frac{1 + \frac{\lambda \Delta t}{2}}{1 - \frac{\lambda \Delta t}{2}} = \left(1 + \frac{\lambda \Delta t}{2}\right)\left(1 + \frac{\lambda \Delta t}{2} + \left(\frac{\lambda \Delta t}{2}\right)^2 + \left(\frac{\lambda \Delta t}{2}\right)^3 + \cdots\right)$$

$\frac{\lambda \Delta t}{2} \neq 1$

$$= \underbrace{1 + \lambda \Delta t + \frac{(\lambda \Delta t)^2}{2}}_{\sim\, e^{\lambda \Delta t}} + \frac{(\lambda \Delta t)^3}{4} + \cdots$$

→ exact soln

up to $O(\Delta t^2)$

This illustrates again that the method is 2^{nd}-order accurate.

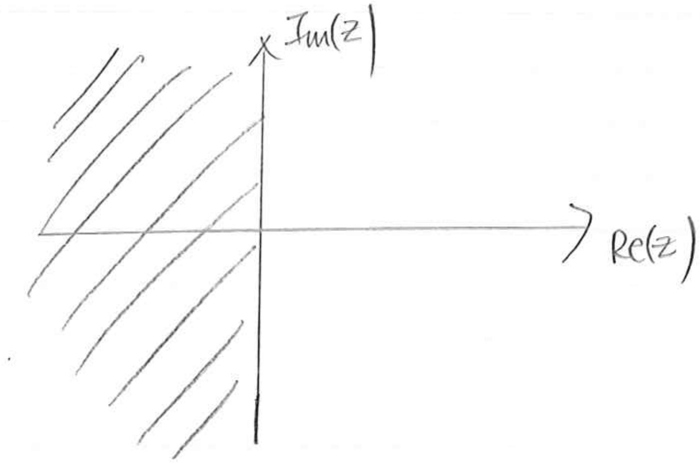
Rmk The stability region for the trapezoidal method can be found if we let

$$z = \lambda \Delta t \quad \in \mathbb{C},$$

As we saw already, the method is stable if

$$Re(z) < 0,$$

Hence the stability region on $\mathbb{C}$ becomes



If $z \in \mathbb{R}$, then $(-\infty, 0)$.

Ex. Backward Euler (BE)

Again we consider the same IVP.

$\rightarrow U^{n+1} = U^n + \Delta t \lambda U^{n+1}$

$\rightarrow (1-\lambda\Delta t) U^{n+1} = U^n$

$$\rightarrow U^{n+1} = \frac{1}{1-\lambda\Delta t} U^n$$

$$= \cdots$$

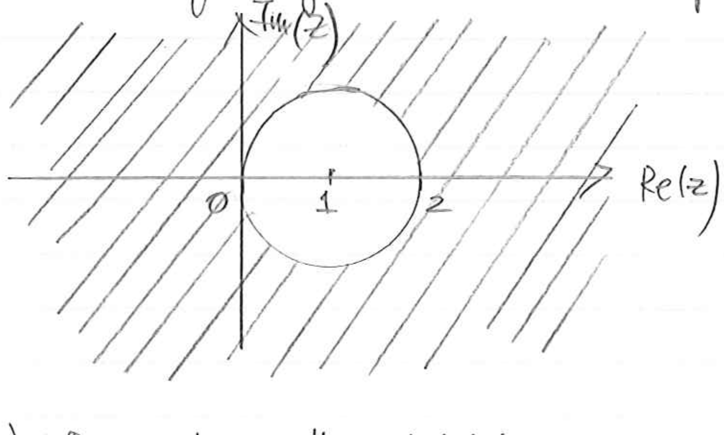
$$= \left(\frac{1}{1-\lambda\Delta t}\right)^n$$

$\rightarrow$ The method is stable if $\left|\frac{1}{1-\lambda\Delta t}\right| \leq 1$

$\rightarrow |1-\lambda\Delta t| \geq 1 \quad \cdots \quad (5)$

$\rightarrow$ If we let $\lambda \in \mathbb{C}$ with $Re(\lambda) < 0$, and let $z = \lambda\Delta t \in \mathbb{C}$, then

(5) is equivalent to $1 \leq |1-z|$, giving the stability region in a complex plane:



$\rightarrow$ If $\lambda \in \mathbb{R}$, then the stability region becomes $(-\infty, 0)$. with $\lambda < 0$.

Rmk. Again, we see that

$$\frac{1}{1-\lambda\Delta t} = \underbrace{1+\lambda\Delta t}_{\sim e^{\lambda\Delta t}} + (\lambda\Delta t)^2 + \cdots$$

which agrees to $e^{\lambda\Delta t}$ up to $\mathcal{O}(\Delta t)$,

Hence confirming the method is 1st order.

Ex. Now consider the <u>Forward Euler's method</u> (FE) applied to the same problem:

$$\begin{cases} u'(t) = \lambda u(t) \\ u(0) = 1. \end{cases} \Rightarrow u(t) = e^{\lambda t} \text{ ; exact soln}$$

Again assume $\begin{cases} \lambda < 0 & \text{if } \lambda \in \mathbb{R}, \text{ or} \\ Re(\lambda) < 0 & \text{if } \lambda \in \mathbb{C}. \end{cases}$

The Forward Euler's method:

$$\begin{aligned} U^{n+1} &= U^n + \Delta t \lambda U^n \\ &= (1 + \lambda \Delta t) U^n \\ &= \cdots \\ &= (1 + \lambda \Delta t)^n \overbrace{U^0}^{=1} \quad (= r^n) \end{aligned}$$

The method is stable if the growth factor:

$$|r| = |1 + \lambda \Delta t| \leq 1, \quad \text{--- (4)}$$

otherwise, it is unstable, although the exact soln is stable.

If $\lambda \in \mathbb{R}$ with $\lambda < 0$, then (4) is equivalent to

$$-2 \leq \lambda \Delta t \leq 0$$

$$\Rightarrow \quad \boxed{0 \leq \Delta t \leq \frac{-2}{\lambda}} \quad (\lambda < 0)$$

the stability bound of Δt.

Rmk. Similarly, the growth factor

$$r = 1 + \lambda \Delta t$$

agrees to the exact soln

$$e^{\lambda \Delta t} = 1 + \lambda \Delta t + \frac{(\lambda \Delta t)^2}{2} + \frac{(\lambda \Delta t)^3}{3!} + \cdots$$

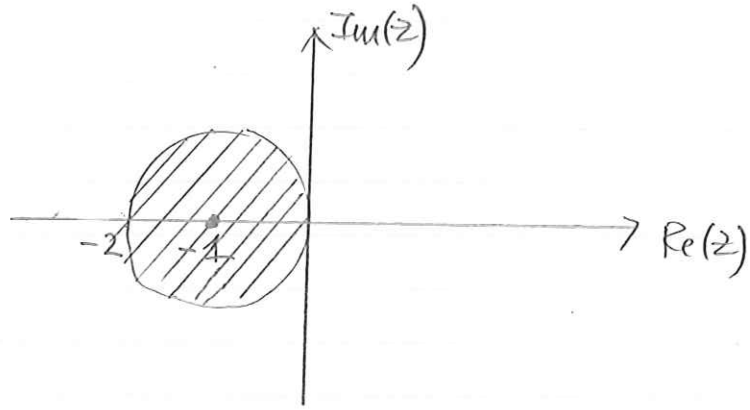
up to $O(\Delta t)$.

$\rightarrow$ Thus the method is again 1st-order method.

Rmk. The stability region is, letting $z = \lambda \Delta t$, and from ④,

$$|1 + z| \leq 1$$

$\Rightarrow$



If $z \in \mathbb{R}$, then $-2 \leq \lambda \Delta t \leq 0$, $\lambda < 0$, as we saw before.

Rmk.

(1) For the trapezoidal & BE methods, we see that they are stable for all choices of Δt
→ a property of implicit methods
→ unconditionally stable

(2) For FE, there is a condition on Δt for stability
→ a property of explicit methods
→ Conditionally stable

Def. the region in the complex plane that determines the stability condition on Δt is called the

absolute stability region,

or simply, stability region.

Rmk. Choice of Δt

- The time step Δt must be chosen small enough to allow the local truncation error Σ_T^{n+1} is small.

- The appropriate step size Δt is different for different methods.

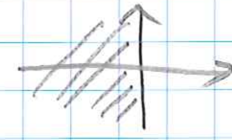
- the size of Δt can make the accuracy of a given method different.

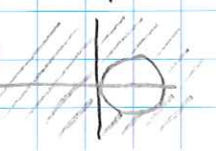
- The size of Δt must be small enough to make the scheme absolutely stable on a given IVP, i.e., we need to choose and check if Δt belongs to the absolute stability region.

Def. An ODE method is said to be A-stable (different from absolute stable) if its region of absolute stability contains the entire left hand plane

$$\{ z \in \mathbb{C} \mid Re(z) \le 0 \}.$$

Ex. Consider the IVP $\begin{cases} u'(t) = \lambda u(t), & \lambda < 0, \\ u(0) = u^0. \end{cases}$

$\rightarrow$ ① Trapezoidal method is A-stable $\rightarrow$ 

② Backward Euler method is A-stable $\rightarrow$ 

 $\leftarrow$ ③ Forward Euler method is NOT A-stable,

Rmk. If Δt is chosen from the absolute stability region, the numerical method and its actual computation v^n becomes

$$\begin{cases} \text{① stable}, & \& \\ \text{② convergent to a correct soln.} \end{cases}$$

Def. The stability poly. π can be expressed in terms of the char. polynomials ρ & σ for LMM:

$$\boxed{\pi(\zeta; z) \equiv \rho(\zeta) - z\,\sigma(\zeta)}$$

Def. The region of absolute stability for LMM is the set of points $z \in \mathbb{C}$ for which

$\pi(\zeta; z)$ satisfies the root condition,

i.e., (i) $|\zeta_j| \le 1$, for distinct roots, $j = 1, \cdots, r$,

(ii) $|\zeta_n| \lneq 1$ if ζ_n is a repeated root of π.

Ex. Forward Euler's method :

$$U^{n+1} = U^n + \Delta t\, f(U^n)$$

$$\Rightarrow \begin{cases} \rho(\zeta) = \zeta - 1 \\ \sigma(\zeta) = 1 \end{cases}$$

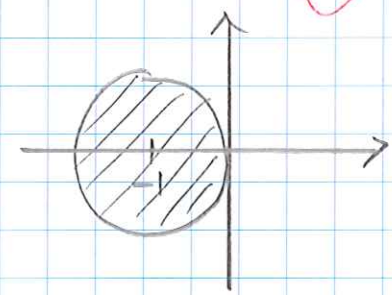
$$\Rightarrow \pi(\zeta; z) = \zeta - 1 - z = \zeta - (1+z).$$

$\Rightarrow$ π has the single root $\zeta_1 = 1+z$.

$\Rightarrow$ The root condition implies to have

$$|\zeta_1| \le 1 \iff |1+z| \le 1$$

$\Rightarrow$



Ex. Backward Euler's method :

$$U^{n+1} = U^n + \Delta t\, f(U^{n+1})$$

$$\Rightarrow \begin{cases} \rho(\zeta) = \zeta - 1 \\ \sigma(\zeta) = \zeta \end{cases}$$

$$\Rightarrow \pi(\zeta; z) = \zeta - 1 - z\zeta = (1-z)\zeta - 1$$

$\Rightarrow$ π has the single root $\zeta_1 = \dfrac{1}{1-z}$

$\Rightarrow$ The root condition gives

$$|\zeta_1| \le 1 \iff |1-z| \ge 1$$

$\Rightarrow$

Ex. Trapezoidal method:

$$U^{n+1} - U^{n} = \frac{\Delta t}{2}\left[f(U^{n}) + f(U^{n+1})\right]$$

$$\Rightarrow \begin{cases} \rho(\zeta) = \zeta - 1 \\ \sigma(\zeta) = \frac{1}{2}(\zeta + 1) \end{cases}$$

$$\Rightarrow \pi(\zeta; z) = \zeta - 1 - \frac{z}{2}(\zeta + 1)$$

$$= \left(1 - \frac{z}{2}\right)\zeta - \left(1 + \frac{z}{2}\right)$$

$\Rightarrow$ The simple root $\zeta_1 = \dfrac{1 + z/2}{1 - z/2}$

$$\Rightarrow |\zeta_1| \leq 1 \iff Re(z) \leq 0$$

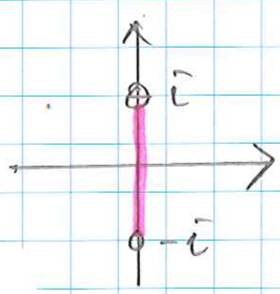
Ex. Midpoint method:

$$U^{n+2} - U^{n} = 2\Delta t\, f(U^{n+1})$$

$$\Rightarrow \begin{cases} \rho(\zeta) = \zeta^2 - 1 \\ \sigma(\zeta) = 2\zeta \end{cases}$$

$$\Rightarrow \pi(\zeta; z) = (\zeta^2 - 1) - z(2\zeta) = \zeta^2 - 2z\zeta - 1$$

$$\Rightarrow \zeta_{1,2} = z \pm \sqrt{z^2 + 1}$$



$\Rightarrow$ If $z = \alpha i$ with $|\alpha| < 1$, then $|\zeta_1| = |\zeta_2| = 1$, $\zeta_1 \neq \zeta_2$, the root condition is satisfied. ✓

$\Rightarrow$ Otherwise, the root condition is NOT satisfied, for example, if $z = \pm i$, then $\zeta_1 = \zeta_2$, a repeated root with $|\zeta_1| = |\zeta_2| = 1$. ⊗