§1.8 Zero-stability and convergence for IVP

- The goal we want to achieve is:

$$\boxed{\text{Consistency} + \text{stability} = \text{convergence}}$$

- To provide a simpler version of the consistency conditions for both one-step methods and LMM, let's rewrite the general form of LMM:

(A)
$$\sum_{j=0}^{r} \alpha_j U^{n+j} = \Delta t \sum_{j=0}^{r} \beta_j f(t^{n+j}, U^{n+j}),$$

Consistency conditions:

(a) $\rho(1) = 0$
(b) $\rho'(1) = \sigma(1)$

in a different form:

(A')
$$U^{n+r} = \alpha_{r-1} U^{n+r-1} + \alpha_{r-2} U^{n+r-2} + \cdots + \alpha_0 U^n + \Delta t\, \psi(t^{n+r}, t^{n+r-1}, \cdots, t^n; \underbrace{U^{n+r}, U^{n+r-1}, \cdots; U^n}_{\text{big } U\text{'s.}}; \Delta t)$$

Consistency conditions:

(a) $\rho(1) = 0$
(b') $\psi(t, t, \cdots t; \overbrace{u, u, \cdots; u}^{\text{small } u\text{'s}}; 0) = \rho'(1) f(t, u)$,

where
$$\psi(t^{n+r}, t^{n+r-1}, \cdots; t^n; U^{n+r}, U^{n+r-1}, \cdots, U^n; \Delta t) \equiv \sum_{j=0}^{r} \beta_j f(t^{n+j}, U^{n+j})$$

- It is easy to show that (b) & (b') are equivalent:

(pf) Let $\psi(t,\cdots,t;u,\cdots,u;0) = \rho'(1) f(t,u)$

$\Leftrightarrow [\beta_0+\beta_1+\cdots+\beta_r] f(t,u) = \rho'(1) f(t,u)$

$$\Leftrightarrow \underbrace{\sum_{j=0}^{r} \beta_j}_{\textstyle \| \atop \textstyle \sigma(1)} = \rho'(1)$$

∎

- Using (A'), we write a general explicit one-step method

$$U^{n+1} = U^n + \Delta t\, \psi(t^n, U^n, \Delta t), \text{ where}$$

we assume ψ is

(i) continuous in t & Δt, &
(ii) Lipschitz continuous in u, say with the Lipschitz constant L.

- We note that all one-step methods have, for the characteristic polynomial ρ:

$$\rho(\zeta) = \zeta - 1,$$

and hence $\rho(1) = 0$ & $\rho'(\zeta) = 1$ always.

Rmk. We can now say that the one-step method is <u>consistent</u> if

$$\boxed{\psi(t;u;0) = f(t,u)}$$

- Note that the following one-step methods are consistent:
  (1) RK (r-stage)
  (2) Euler (forward, backward)
  (3) trapezoidal,

and the following LMM are consistent
  (4) Adams – Bashforth
  (5) Adams – Moulton.

- We hope to say that if a numerical method is <u>consistent</u> then the method is <u>convergent</u>.

- Notice here that we don't even explicitly mention anything on <u>stability</u>.

- Regardless, we are going to show that <u>all consistent one-step methods</u> turn out to be <u>convergent</u>, but this is not necessarily true for LMM.

- Let's see why this is true for one-step methods, and also see what else is needed for LMM for convergence.

Thm. If a general explicit one-step method of the form

$$U^{n+1} = U^n + \Delta t\, \psi(t^n, U^n, \Delta t) \quad \cdots \text{①}$$

is consistent, then it is convergent.

Proof. Note that the local truncation error is given by

$$E_{LT}^{n+1} = \frac{1}{\Delta t}\left[u(t^{n+1}) - u(t^n)\right] - \psi(t^n, u^n, \Delta t).$$

Rewriting it, we get the exact relation:

$$u(t^{n+1}) = u(t^n) + \Delta t\, \psi(t^n, u^n, \Delta t) + \Delta t\, E_{LT}^{n+1}. \quad \cdots \text{②}$$

Then

$$\begin{aligned} E_g^{n+1} &= U^{n+1} - u(t^{n+1}) \\ &= \left[U^n + \Delta t\, \psi(t^n, U^n, \Delta t)\right] \quad \leftarrow \text{①} \\ &\quad - \left[u(t^n) + \Delta t\, \psi(t^n, u^n, \Delta t) + \Delta t\, E_{LT}^{n+1}\right] \quad \leftarrow \text{②} \\ &= \underbrace{\left(U^n - u(t^n)\right)}_{(= E_g^n)} + \Delta t\left(\psi(t^n, U^n, \Delta t) - \psi(t^n, u^n, \Delta t)\right) \\ &\quad - \Delta t\, E_{LT}^{n+1} \end{aligned}$$

$\Rightarrow$ Using the Lipschitz continuity, we obtain

$|1 + \Delta t L| \le e^{|L|\Delta t}$

$$\begin{aligned} |E_g^{n+1}| &\le |E_g^n| + \Delta t\, L\, |E_g^n| + \Delta t\, |E_{LT}^{n+1}| \\ &\le |1 + \Delta t L|\, |E_g^n| + \Delta t\, |E_{LT}^{n+1}| \\ &\le e^{|L|\Delta t}\, |E_g^n| + \Delta t\, |E_{LT}^{n+1}| \quad \cdots \text{③} \end{aligned}$$

We also consider a finite time interval $0 \le t \le T$, so that $t_n = n\Delta t \le T$, for each T.

Then ③ $\le \cdots$

$$\le e^{|L| n\Delta t} |E_g^0| + \Delta t \sum_{m=1}^{n} (1+\Delta t L)^{n-m} |E_{LT}^{m-1}| \quad \text{--- ④}$$

Noting
$$\begin{aligned} (1+\Delta t L)^{n-m} &\le e^{(n-m)|L|\Delta t} \\ &\le e^{|L| n\Delta t} \\ &\le e^{|L|T}, \end{aligned}$$

and we continue from ④:

$$④ \le e^{|L|T} |E_g^0| + \sum_{m=1}^{n} e^{|L|T} |E_{LT}^{m-1}|$$

$$\le e^{|L|T} \left(|E_g^0| + \Delta t\, n \max_{1\le m\le n} |E_{LT}^{m-1}| \right)$$

$|E_g^0| = 0$ & $\|E_{LT}\|_\infty \equiv \max_{1\le m\le n} |E_{LT}^{m-1}|$

$$\le \underbrace{e^{|L|T} \cdot T}_{\text{bdd}} \cdot \underbrace{\|E_{LT}\|_\infty}_{\to 0}, \text{ since consistent}$$

$$\to 0.$$

Therefore, any <u>consistent one-step method</u> is <u>convergent</u>. ▨

- Now we show that, for LMM, consistency is <u>NOT</u> sufficient for convergence.

- To proceed further, we first review how to solve <u>linear difference equations</u> in general.

<u>Thm</u>. Consider the general homogeneous linear difference eqn,

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = 0, \quad \cdots (5)$$

The general soln of this eqn. is given as

$$U^n = \zeta^n, \quad \text{where}$$

ζ is a root of the polynomial

$$\rho(\zeta) = \sum_{j=0}^{r} \alpha_j \zeta^j. \quad \cdots (6)$$

<u>proof</u>. Plugging $U^n = \zeta^n$ into (5), we get

$$\sum_{j=0}^{r} \alpha_j \zeta^{n+j} = 0.$$

Dividing by ζ^n, we obtain

$$\sum_{j=0}^{r} \alpha_j \zeta^n = 0, \quad \text{which implies that}$$

ζ is a root of (5).

We note that (6) is just the first characteristic poly. of LMM for the consistency conditions. ▨

Cor. In general, if $\rho(\zeta)$ has r roots, $\zeta_1, \cdots, \zeta_r$:

$$\rho(\zeta) = \alpha_r(\zeta-\zeta_1)(\zeta-\zeta_2)\cdots(\zeta-\zeta_r), \text{ then}$$

Since the difference eqn (5) is linear, any linear combination of the solns $\zeta_1, \cdots, \zeta_r$

$$U^n = C_1\zeta_1^n + C_2\zeta_2^n + \cdots + C_r\zeta_r^n$$

is also a soln for distinct $\zeta_1, \cdots, \zeta_r$.

If the roots are not distinct, say, $\zeta_1 = \zeta_2$, then ζ_1^n & ζ_2^n are not linearly independent, and the $r \times r$ linear system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \zeta_1 & \zeta_2 & \cdots & \zeta_r \\ \zeta_1^2 & \zeta_2^2 & \cdots & \zeta_r^2 \\ \vdots & & & \\ \zeta_1^{r-1} & \zeta_2^{r-1} & \cdots & \zeta_r^{r-1} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_r \end{bmatrix} = \begin{bmatrix} U^0 \\ U^1 \\ U^3 \\ \vdots \\ U^{r-1} \end{bmatrix}$$

becomes singular. In this case, we fix the issue easily by noting that

$n\zeta_1^n$ is also a soln.

Therefore, the general soln with $\zeta_1 = \zeta_2$ becomes

$$U^n = C_1\zeta_1^n + C_2 n\zeta_1^n + C_3\zeta_3^n + \cdots + C_r\zeta_r^n,$$

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Ex 1 Consider a two-step method
$$U^{n+2} = 4U^{n+1} - 3U^{n} - 2\Delta t\, f(t^n, U^n).$$

(1) Consistency check:

$$\rho(\zeta) = \zeta^2 - 4\zeta + 3$$

$$\Rightarrow \text{(i) } \rho(1) = 0 \quad \checkmark$$
$$\text{(ii) } \rho'(\zeta) = 2\zeta - 4 \quad \& \quad \sigma(\zeta) = -2,$$
$$\therefore \rho'(1) = -2 = \sigma(1). \quad \checkmark$$

OK, one can instead show

$$\text{(ii)}' \ \psi(t; u; 0) = -2 f(t,u) = \rho'(1) f(t,u) \ \checkmark$$

$\therefore$ The method is consistent.

(2) Now we consider applying the method to the IVP:
$$\begin{cases} u' = u = f(t,u) \\ u(0) = 1. \end{cases}$$

$$\Rightarrow \begin{cases} U^{n+2} = 4U^{n+1} - 3U^n - 2\Delta t\, U^n \\ \quad\quad\;\; = 4U^{n+1} - (3 + 2\Delta t) U^n \quad \cdots \ 7 \\ U^0 = 1, \text{ and let's assume } U^1 = \text{known}. \end{cases}$$

We find the characteristic poly $\rho(\zeta)$ for ⑦:

$$\rho(\zeta) = \zeta^2 - 4\zeta + (3+2\Delta t) = 0$$

$$\Rightarrow \zeta = 2 \pm \sqrt{4-(3+2\Delta t)} = 2 \pm \sqrt{1-2\Delta t}$$

$\Rightarrow$ The general soln becomes

$$U^n = c_1 \left(2-\sqrt{1-2\Delta t}\right)^n + c_2 \left(2+\sqrt{1-2\Delta t}\right)^n$$

$\Rightarrow$ Note that

$$\begin{cases} \left|2-\sqrt{1-2\Delta t}\right| \le 1, & \& \\ \left|2+\sqrt{1-2\Delta t}\right| > 1 & . \end{cases}$$

$\Rightarrow U^n \to \infty$ as $n \to \infty$.

$\Rightarrow$ This suggests that the method is not convergent although it is consistent.

Ex 2. Consider the consistent LMM given by

$$U^{n+2} - 2U^{n+1} + U^n = \frac{\Delta t}{2}\left(f(U^{n+2}) - f(U^n)\right)$$

and use it to solve the IVP

$$\begin{cases} u'(t) = 0 = f(t,u) \\ u(0) = U^0, \quad u(t^1) = U^1. \end{cases}$$

Then we have the difference eqn:

$$U^{n+2} - 2U^{n+1} + U^n = 0,$$ which has

note that → $\rho(1) = 0$, and $\rho'(1) = 0 = \sigma(1)$, where $\sigma(\zeta) = \frac{1}{2}(\zeta^2 - 1)$ ∴ the method is consistent.

the char. eqn: $\rho(\zeta) = \zeta^2 - 2\zeta + 1 = 0$,

⇒ $\zeta = 1$ is the repeated root.

⇒ The general soln is given as

$$U^n = c_1 \cdot 1 + c_2 \cdot n \cdot 1$$

⇒ If using the ICs to determine c_1 & c_2:

$$\begin{cases} U^0 = c_1 \\ U^1 = c_1 + c_2 = U^0 + c_2 \Rightarrow c_2 = U^1 - U^0 \end{cases}$$

$$\Rightarrow \quad U^n = U^0 + (U^1 - U^0)\,n.$$

⇒ Again, $U^n \to \infty$ as $n \to \infty$.

Ex 3. Consider the consistent LMM :

$$U^{n+3} - 2U^{n+2} + \frac{5}{4}U^{n+1} - \frac{1}{4}U^{n} = \frac{\Delta t}{4} f(U^{n}).$$

(1) $\begin{cases} \rho(\zeta) = \zeta^3 - 2\zeta^2 + \frac{5}{4}\zeta - \frac{1}{4} \\ \sigma(\zeta) = \frac{1}{4} \end{cases}$

$\Rightarrow \begin{cases} \rho(1) = 0 \quad \checkmark \\ \rho'(\zeta) = 3\zeta^2 - 4\zeta + \frac{5}{4}, \end{cases}$

$\therefore \rho'(1) = \frac{1}{4} = \sigma(1) \quad \checkmark$

$\therefore$ the method is consistent.

(2) $\rho(\zeta) = (\zeta - 1)(\zeta - \frac{1}{2})^2$

$\Rightarrow U^n = c_1 + c_2\left(\frac{1}{2}\right)^n + c_3 n \left(\frac{1}{2}\right)^n$

$\Rightarrow$ Here, the repeated root $\zeta = \frac{1}{2}$ has modulus less than 1, and as a result,
U^n is convergent.

Rmk. From Ex 1, we see that if $\rho(\zeta)$ has distinct roots, they are required to be

$$|\zeta_j| \le 1, \quad j=1,\cdots,r, \text{ for convergence.}$$

Rmk. From Ex 2 & 3, we additionally see that in case that $\rho(\zeta)$ has any repeated root ζ_n, then the repeated root ζ_n need to satisfy

$$|\zeta_n| \lneqq 1, \quad \text{for convergence.}$$

Rmk. We are now ready to state additional stability condition, called zero-stability, in order to avoid the numerical soln $U^n \to \infty$ as $n \to \infty$.

Def. An r-step LMM is said to be zero-stable if the roots of the char. poly $\rho(\zeta)$ satisfy the following conditions;

(i) $|\zeta_j| \le 1, \quad j=1,\cdots,r$ (distinct roots)

If there is any repeated root, say, ζ_n, then

(ii) $|\zeta_n| \lneqq 1$.

$\Rightarrow$ (i) & (ii) ; the root condition for the char. poly ρ.

Rmk. It is easy to see that all one-step methods are zero-stable.

Question: Is zero-stability good enough for convergent?

Answer Yes!

Thm (Dahlquist) For LMMs applied to the IVP

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(0) = u^0 \end{cases},$$

$$\boxed{\text{consistency} + \text{zero-stability} \iff \text{convergence}}$$

Question: Are we done now?

Answer: No! Although a consistent zero-stable method is convergent, it may have other stability problems that show up if Δt is chosen to be too large in actual computation.

Therefore, when applied to an actual programming practice, the numerical soln is not useful in such a case. (i.e., unstably inaccurate)

Remember that zero-stability only guarantees that the numerical soln v^n does not blow up as $n \to \infty$, and this doesn't mean that v^n is mathematically correct for all choices of Δt.