§1.5 Errors, consistency, stability & convegence

Recall that U^n is the numerical approximations to the exact solutions $u(t^n)$ at $t=t^n$, $n \geq 0$.

Def. Let D^n be the exact solution of the discrete difference equations (DE) of a given IVP, e.g.,

D^n : exact soln of the forward Euler's method

$$\Rightarrow \frac{D^{n+1} - D^n}{\Delta t} = f(D^n)$$

$\Rightarrow D^n$ satisfies this eqn without producing any errors in computer arithmetic.

- When we study numerical solns of ODEs & PDEs, the solns are affected by numerical errors.
- They mainly come from the two sources of numerical errors:
  - (i) discretization error,
  - (ii) round-off error

Def. The discretization error E_d^n at $t = t^n$ is defined by

$$\boxed{E_d^n = u^n - D^n}$$

exact for DE.

exact for IVP

Def. The round-off error E_r^n at $t = t^n$ is defined by

$$\boxed{E_r^n = D^n - U^n}$$

exact for DE

appn to u^n.

(repeated computer arithmetic operation due to rounding off numbers to some significant digits)

Def. The global error E_g^n at $t = t^n$ is defined by

$$\boxed{\begin{aligned} E_g^n &= E_d^n + E_r^n \\ &= u^n - U^n \end{aligned}}$$

Def. We say that the numerical method is convergent at $t = t^n$ in a given norm $\|\cdot\|$ if

$$\boxed{\lim_{\Delta t \to 0} \| E_g^n \| = 0}$$

Def. Denote N by the (linear) <u>numerical operator</u> mapping the approximate soln at one time step to the approximate soln at the next time step.

Then a general explicit numerical method can be written as

$$U^{n+1} = N(U^n).$$

Def. We define the <u>one-step error</u> E_{1step}^{n+1} by

$$\boxed{E_{1step}^{n+1} = u^{n+1} - N(u^n)}.$$

Rmk. Note that what we measure here is the following:

(i) evolve the nth step to $(n+1)$ <u>numerically</u> assuming that we know the <u>exact</u> solution at $t = t^n$, i.e., u^n.
$\Rightarrow$ this term is $N(u^n)$.

(ii) Compare $N(u^n)$ with the <u>exact</u> soln u^{n+1} at $t = t^{n+1}$.

Def. The <u>local truncation (LT) error</u> E_{LT}^{n+1} is given by averaging E_{1step}^{n+1} over Δt:

$$\boxed{E_{LT}^{n+1} = \frac{1}{\Delta t} E_{1step}^{n+1}}$$

Def. We say that the numerical method is <u>of pth order</u> (or <u>pth order accurate</u>) in time if for all sufficiently smooth data with compact support, the local truncation error is given by

$$\boxed{E_{LT}^n = \mathcal{O}(\Delta t^p)}$$

Def. We say the numerical method is <u>consistent</u> in $\|\cdot\|$ with a differential eqn. if

$$\boxed{\lim_{\Delta t \to 0} \| E_{LT}^n \| = 0}$$

for all smooth fns $f(t, u(t))$ that satisfies the given ODE.

Def. We say the (linear) numerical method defined by the linear operator N is <u>stable</u> in $\|\cdot\|$ if $\exists$ C s.t., for each T,

$$\boxed{\| N^n \| \le C}, \quad \forall\, t^n = n\Delta t \le T.$$

$\Rightarrow$ That is to say, the nth power of the operator N is uniformly bdd. up to this time T.

$\Rightarrow$ If a method is stable, it means that the local errors do not grow catastrophically and hence a bound on the global error can be obtained in terms of the local errors.

Rmk. In particular, the linear numerical method is stable if
$$\|N\| \leq 1, \cdots ①$$

proof. By the Cauchy-Schwarz inequality, we get
$$\|N^n\| \leq \|N\|^n \leq 1.$$

Rmk. One can also say that the method is stable if
$$\|U^{n+1}\| \leq \|U^n\|, \text{ for all } n. \cdots ②$$

proof. Recalling $U^{n+1} = N(U^n)$, we see that
$$\frac{\|N(U^n)\|}{\|U^n\|} = \frac{\|U^{n+1}\|}{\|U^n\|} \leq 1, \ \forall n, \ \|U^n\| \neq 0. \cdots ③$$

Since ③ is true for all U^n, taking sup to get
$$\sup_{U \neq 0} \frac{\|N(U)\|}{\|U\|} \leq 1,$$

which gives
$$\|N\| \leq 1.$$

Rmk The condition ② is stronger than ①.

Rmk. There is a very important relationship among the three important properties :

(i) consistency ($\|E_T^n\| \to 0$ as $\Delta t \to 0$)
(ii) stability ($\|N\| \le 1$)
(iii) convergence ($\|E_g^n\| \to 0$ as $\Delta t \to 0$)

$\Rightarrow$ The fundamental theorem of numerical methods for difference equations ;

$$\text{consistency} + \text{stability} \Leftrightarrow \text{convergence}$$

$\Rightarrow$ For ODEs : Dahlquist's equivalence Thm
For PDEs ; The Lax equivalence Thm for <u>linear</u> PDEs

Rmk. Why is this thm important? To say the method is convergence, one needs to know the exact soln u^n which is not available in most cases. Instead, one can rely on the local numerical properties (consistency & stability) to show convergence indirectly.

Def [Big-Oh] $\mathcal{O}$
If f and g are two fns, and we say <u>f is of order g as $t \to 0$</u>, denoted by

$$f(t) = \mathcal{O}(g(t)) \quad \text{as} \quad t \to 0$$

if $\exists\ C$ s.t

$$|f(t)| \le C\,|g(t)|, \quad \forall t : \text{sufficiently small}$$

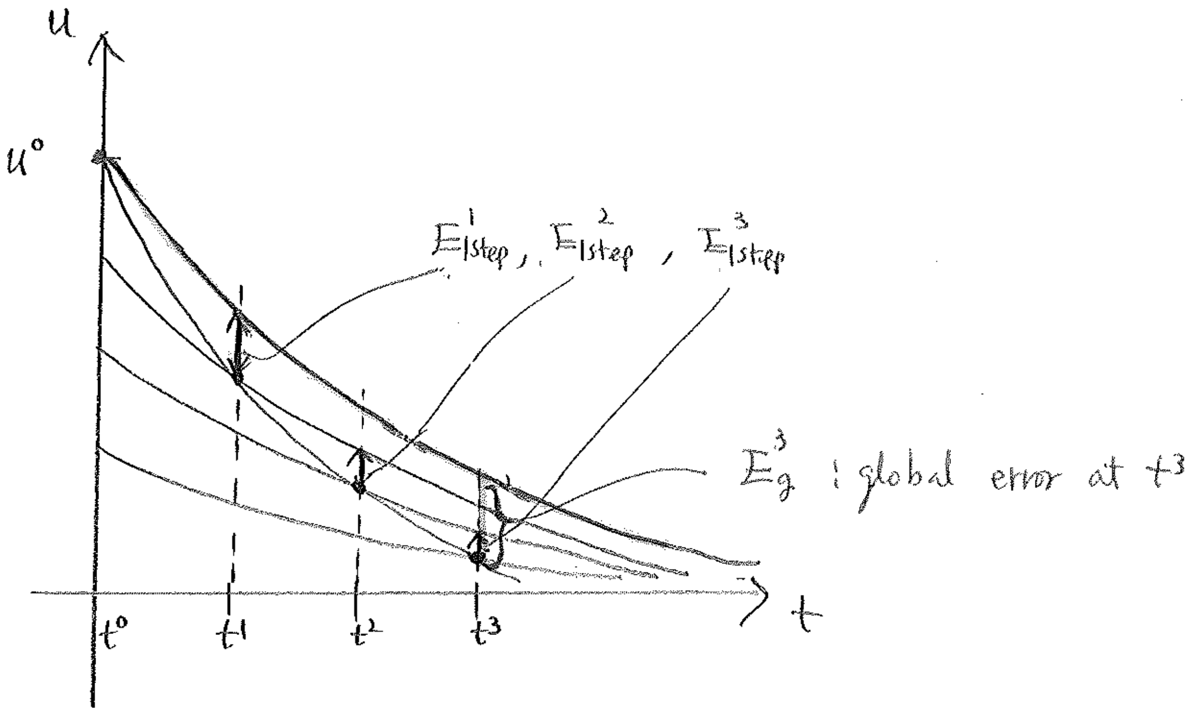
$\Rightarrow$ $f(t)$ decays to zero at least as fast as $g(t)$ does.

Def. [little-oh] o

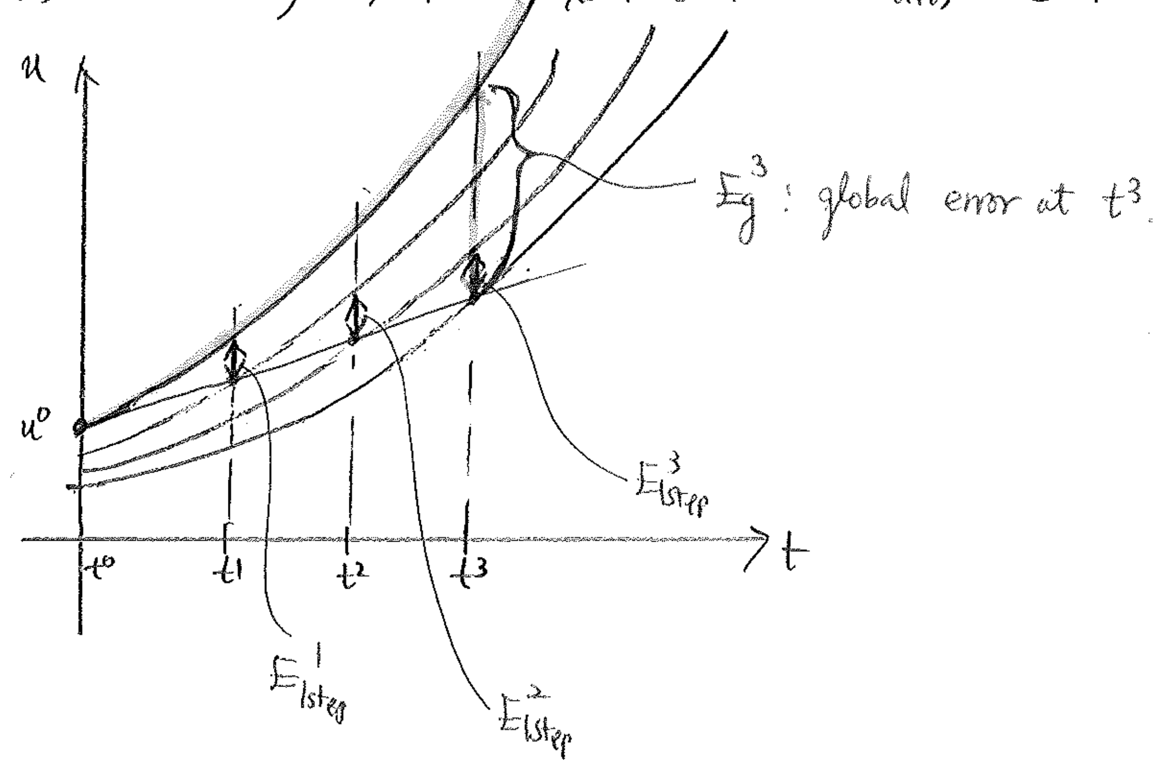
We also write $f(t) = o(g(t))$ as $t \to 0$
if $\left|\frac{f(t)}{g(t)}\right| \to 0$ as $t \to 0$.

$\Rightarrow$ This is stronger than $f(t) = O(g(t))$ and it means $f(t)$ decays to zero faster than $g(t)$ does.

Ex. $u'(t) = -u(t) \Rightarrow$ the exact soln is $u(t) = e^{-t}$



Ex. $u'(t) = u(t) \Rightarrow$ the exact soln is $u(t) = e^t$.



Ex. Show that the leap-frog method is 2^{nd}-order:

Recall $U^{n+1} = U^{n-1} + 2\Delta t f(U^n) \left(= N(U^n)\right)$, $n \geq 1$

(1) First compute the one-step error E_{1step}^{n+1}:

$$E_{1step}^{n+1} = u^{n+1} - N(u^n), \text{ where}$$

$$\begin{aligned} N(u^n) &= u^{n-1} + 2\Delta t f(u^n) \\ &= u^{n-1} + 2\Delta t\, u'(t^n), \end{aligned}$$

$$\Rightarrow E_{1step}^{n+1} = u^{n+1} - \left(u^{n-1} + 2\Delta t\, u'(t^n)\right)$$

Using a Taylor series exansion, we get:

(i) $$\begin{aligned} u^{n+1} &= u(t^{n+1}) \\ &= u(t^n + \Delta t) \\ &= u(t^n) + \Delta t\, u'(t^n) + \frac{\Delta t^2}{2!} u''(t^n) + \frac{\Delta t^3}{3!} u'''(t^n) + \cdots \end{aligned}$$

(ii) $$\begin{aligned} u^{n-1} &= u(t^{n-1}) \\ &= u(t^n - \Delta t) \\ &= u(t^n) - \Delta t\, u'(t^n) + \frac{\Delta t^2}{2!} u''(t^n) - \frac{\Delta t^3}{3!} u'''(t^n) + \cdots \end{aligned}$$

$$\begin{aligned} \Rightarrow E_{1step}^{n+1} &= 2\Delta t\, u'(t^n) + \frac{\Delta t^3}{3} u'''(t^n) - 2\Delta t\, u'(t^n) \\ &= \frac{\Delta t^3}{3} u'''(t^n) + \mathcal{O}(\Delta t^5) \end{aligned}$$

(2) Therfore, $E_{LT}^{n+1} = \frac{1}{\Delta t} E_{1step}^{n+1} = \mathcal{O}(\Delta t^2)$

$\therefore$ The leapfrog method is <u>2^{nd} order.</u>

Rmk. In the previous example, the global error E_g^{n+1} may not be of 2^{nd}-order accurate.

However, if the method is stable (in some region), then we see that the global error exhibits the same rate of the local truncation error.

Ex. Show that the forward Euler's method is 1^{st}-order.

Recall $\quad U^{n+1} = U^n + \Delta t\, f(U^n) \quad (= N(U^n)) \quad , n \geq 0$

$$(1)\quad E_{1step}^{n+1} = u^{n+1} - N(u^n)$$
$$= u(t^n + \Delta t) - [u(t^n) + \Delta t\, u'(t^n)]$$
$$= \cancel{u(t^n)} + \cancel{\Delta t\, u'(t^n)} + \frac{\Delta t^2}{2} u''(t^n) + \cdots$$
$$- \cancel{u(t^n)} - \cancel{\Delta t\, u'(t^n)}$$
$$= \mathcal{O}(\Delta t^2)$$

$$(2)\quad E_{LT}^{n+1} = \frac{1}{\Delta t} E_{1step}^{n+1} = \mathcal{O}(\Delta t)$$

$\therefore$ 1^{st}-order method.

Ex. Similarly, we can show that the backward Euler's method is 1^{st}-order accurate.