§1.4. Some basic numerical methods

Basic Concepts.

Def. We denote temporal discretizations by

$$\boxed{t^n = n\Delta t}, \quad n = 0, 1, \cdots, M,$$

where Δt is the time step size of the interval over which we seek for numerical solns.

Consider the IVP :

$$\begin{cases} u'(t) = f(u(t)) & \cdots \quad (4) \\ u(t^0) = u^0 . \end{cases}$$

Def. We denote the pointwise values of the exact soln of (4) by

$$\boxed{u^n = u(t^n)} .$$

This is the analytical soln of the IVP, evaluated at a set of discrete points $t^n = n\Delta t$.

Def. We denote the numerical approximations to the discrete exact solutions u^n by

$$v^n,$$

i.e,

$$\underset{\text{numerical}}{v^n} \approx \underset{\text{analytical (or exact)}}{u^n},$$

- In this way, our goal in this course is to seek for various ways to compute

$$v^1, v^2, \cdots, v^n, \cdots$$

each of which approximates the analytic solutions at $t = t^n$, i.e.,

$$v^n \approx u^n = u(t^n),$$

given an IC

$$v^0 = u^0 = u(t^0).$$

- This is called the "time marching method" at $t = t^1, t^2, \cdots$,

Rmk The superscript "n" is reserved for temporal discretization for both ODEs & PDEs.

Rmk On the other hand, we will use the subscripts i, j, k, etc. for spatial discretizations for PDEs.

Def. Forward Euler's Method

The simplest discrete method to discretize the IVP (4) is given by the (forward) Euler's method which replaces

$$u'(t^n) \text{ with } \frac{v^{n+1}-v^n}{\Delta t}, \text{ giving}$$

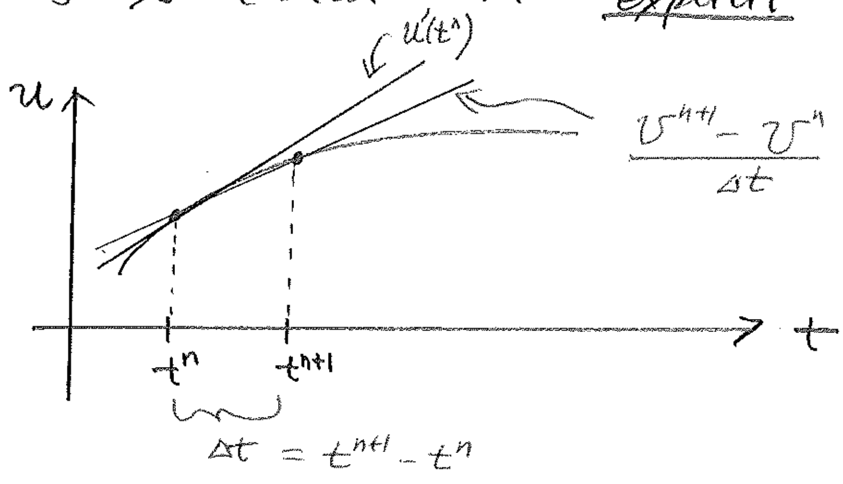
$$\begin{cases} u'(t) = f(u(t)) \\ u(0) = u^0 \end{cases} \Rightarrow \begin{cases} \dfrac{v^{n+1}-v^n}{\Delta t} = f(v^n) \\ v^0 = u^0 \end{cases}$$

$$\Rightarrow \boxed{v^{n+1} = v^n + \Delta t\, f(v^n)}, \quad n = 0, 1, \cdots \quad \cdots (5)$$

⇒ This is the time marching method for the forward Euler's method.

⇒ Note that the RHS of Eq (5) includes only <u>known values</u>, v^n, and they are used to predict a new value v^{n+1}.

⇒ This is called an "<u>explicit</u>" scheme.



Def. Backward Euler's method

On the other hand, if we use a backward in time t, we replace

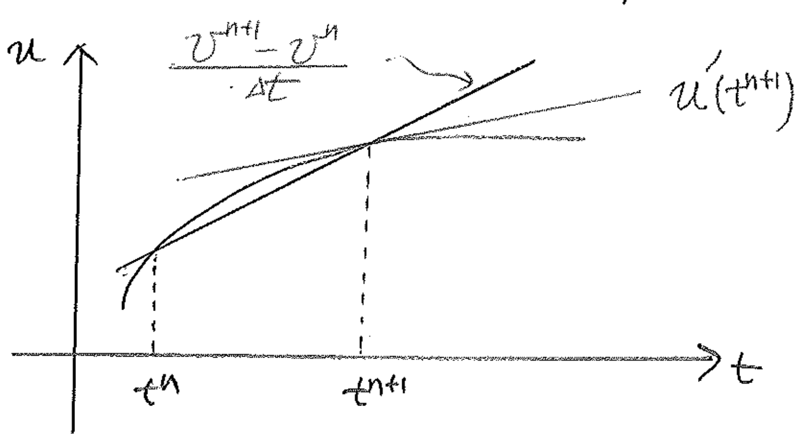
$$u'(t^{n+1}) \quad \text{with} \quad \frac{U^{n+1} - U^n}{\Delta t}, \text{ giving}$$

$$\frac{U^{n+1} - U^n}{\Delta t} = f(U^{n+1}), \text{ or}$$

$$\boxed{U^{n+1} = U^n + \Delta t\, f(U^{n+1})} \quad n = 0, 1, \cdots \quad \cdots (6)$$

⇒ In this case, the RHS includes the unknown value U^{n+1} which is to be solved for, and hence we don't have any information of $f(U^{n+1})$ at the current time $t = t^n$.

⇒ This is called an "implicit" scheme.



Rmk The backward Euler's method must be solved for U^{n+1} using an implicit method that often requires iterative numerical solutions for solution convergence for U^{n+1}. (e.g., Newton's method)

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Def. Trapezoidal method
Another implicit method by averaging the two Euler's methods;

$$\frac{U^{n+1} - U^{n}}{\Delta t} = \frac{1}{2}\left(f(U^{n}) + f(U^{n+1})\right),$$

is called the trapezoidal method.

⇒ This method uses the symmetric approximation which gives a 2nd-order accurate in time

⇒ The two Euler's methods are only 1st-order accurate in time.

⇒ We will define orders of accuracy soon.

Def. A discretization method is called a <u>one-step method</u> if U^{n+1} can be determined from U^n alone, but not U^k, $\forall k<n$.

Def. A discretization method is called a <u>multistep method</u> if U^{n+1} is determined from more than U^n, i.e., U^k, for some $k<n$.

Ex. Forward Euler
Backward Euler
Trapezoidal method
$\Rightarrow$ one-step method

Ex. If we adopt a <u>centered differencing</u> to approximate $u(t^n)$;

$$u(t^n) \approx \frac{u(t^n+\Delta t) - u(t^n-\Delta t)}{2\Delta t} = \frac{U^{n+1} - U^{n-1}}{2\Delta t},$$

we get a multistep method called <u>the midpoint</u> (or <u>leapfrog</u>) method:

$$\boxed{U^{n+1} = U^{n-1} + 2\Delta t\, f(U^n)}, \quad n \geq 1.$$

$\Rightarrow$ This is a <u>second order</u> explicit method.

Rmk. A multistep method provided a higher order method. (Good thing)

Rmk. A multistep method cannot self start and requires additional one-step method(s) to get going for the first few steps. (Bad(!) thing, but not too bad)