Chapter 1. Introduction.

§1.1 Classification of differential equations.

- We consider several different ways of classifying differential equations (DEs).

(1) number of independent variables of the unknown functions

→ ODEs (single) vs. PDEs (multiple)

(2) number of unknown functions

→ single DE vs. a system of DEs

(3) orders

→ nth order DEs

(4) linearity

→ linear DEs vs. nonlinear DEs

Def. The order of a DE is the order of the highest derivative that appears in the equation.

The general form of an kth order ODE is

$$u^{(k)} = f(t, u(t), u'(t), \cdots, u^{(k-1)}(t)) .$$

Ex. $(y')^2 + ty' + 4y = 0$ is a first-order ODE.

Ex. $y''' + 2e^t y'' + yy' = t^4$ is a third-order ODE.

Ex. $u_{xx} + u_{yy} = 0$ is a 2^{nd}-order PDE.
(elliptic PDE)

Ex. $u_t + a u_x = 0$ is a 1^{st}-order PDE.
(hyperbolic PDE, or advection eqn)

Ex. $u_t = k u_{xx}$ is a 2^{nd}-order PDE.
(parabolic PDE, or diffusion eqn, heat eqn)

Ex. $u_t + a u_{xx} = k u_{xx}$ is a 2^{nd}-order PDE.
(mixed type PDE, or advection-diffusion eqn).

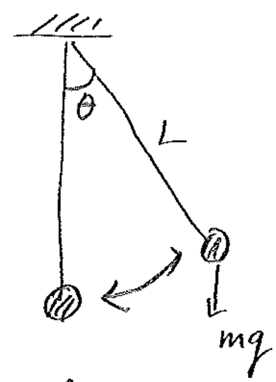
Def. A DE $f(t, u(t), u'(t), \cdots, u^{(k)}(t)) = 0$ is <u>linear</u> if f is a linear function of $u, u', \cdots, u^{(k)}$.

Otherwise, it is said to be <u>nonlinear</u>.

<u>Note</u> There is no dependency on t.

<u>Ex</u>. The general form of a kth-order ODE is
$$a_k(t)\, y^{(k)} + a_{k-1}(t)\, y^{(k-1)} + \cdots + a_0(t)\, y = g(t),$$
where $g(t)$ is a nonhomogeneous term with $g(t) \neq 0$.

<u>Ex</u>. $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\sin\theta = 0$
for an oscillating pendulum is a 2nd-order nonlinear ODE.



<u>Ex</u>. $y' + ty^2 = 0$ is a 1st-order nonlinear ODE.

<u>Ex</u>. $y''' + ty' + (\cos^2 t)\, y = t^3$ is a 3rd-order linear nonhomogeneous ODE.

<u>Ex</u>. $u_t + a u_x = 0$, a: constant, is a 1st-order linear PDE.

Ex. $u_t + uu_x = 0$ is a 1st-order nonlinear PDE

This equation is called the Burgers equation, which can be rewritten as

$$u_t + \left(\frac{u}{2}\right)_x = 0.$$

Ex. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$ is a 4th-order linear PDE.

Ex. $u_t + uu_x = 1 + u_{xx}$ is a 2nd-order nonlinear PDE.

- In this chapter, we study time-dependent ODEs, in particular, initial value problems (IVPs).

- We will consider boundary value problems (BVPs) later.

- IVP takes of the form:

$$\begin{cases} \text{ODE} : u'(t) = f(t, u(t)), \quad t > t_0 \\ \text{IC} : u(t_0) = u_0 \end{cases} \qquad \cdots (1)$$

- Usually, we take $t_0 = 0$

- (1) represents:
  (i) a system of ODEs if u & f are vectors,
  or
  (ii) a scalar ODE if u & f are scalar real variables & functions.

- We mainly focus on studying <u>1st-order</u> ODEs because any higher order ODEs can always converted to an equivalent system of 1st-order ODEs, using a new set of variables:

$$x_1 = u, \ x_2 = u', \ x_3 = u'', \ \cdots, \ x_k = u^{(k-1)}.$$

<u>Ex</u> Given $u'' + 0.125u' + u = 0$, a 2nd-order ODE, let

$$\begin{cases} x_1(t) = u(t), \\ x_2(t) = u'(t). \end{cases}$$

Then $x_1' = u' = x_2$ and $u'' = x_2'$.
Therefore, we can rewrite the ODE into

$$\begin{cases} x_1' = x_2 \\ x_2' = -0.125x_2 + x_1. \end{cases}$$

Def. An ODE is said to be <u>autonomous</u> if f does not depend explicitly on t, i.e.,

$$u' = f(u).$$

Def. An ODE is said to be <u>nonautonomous</u> if f explicitly depends on t, i.e.,

$$u' = f(t, u).$$

Rmk. We often prefer to write ODEs in the autonomous form for simplicity.

Rmk. A nonautonomous ODE can be always converted to an autonomous ODE by introducing an additional dependent variable which replaces t.

Ex. Consider $t^2 u'' + t u' + (t^2 - 0.25) u = 0, \quad t > 0.$

Let $\begin{cases} x_1(t) = t, \\ x_2(t) = u(t) \\ x_3(t) = u'(t) \end{cases}$,

Then, $x_1'(t) = 1,$

$x_2'(t) = u'(t) = x_3(t)$

$x_3'(t) = u''(t).$

Rewriting the equation, after dividing by t^2:

$$u'' = -\frac{1}{t} u' - \left(1 - \frac{1}{4t^2}\right) u$$

$$\Leftrightarrow \quad x_3' = -\frac{x_3}{x_1} - \left(1 - \frac{1}{4x_1^2}\right) x_2$$

In summary, we obtain an equivalent first-order autonomous system:

$$\underset{\sim}{u}'(t) = f(\underset{\sim}{u}(t)), \text{ where}$$

$$\underset{\sim}{u}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},$$

$$f(\underset{\sim}{u}(t)) = \begin{bmatrix} 1 \\ x_3(t) \\ -\dfrac{x_3(t)}{x_1(t)} - x_2(t) + \dfrac{x_2(t)}{4x_1(t)^2} \end{bmatrix}$$

If the original 2^{nd}-order nonautonomous ODE comes with initial conditions (ICs):

$$\begin{cases} u(0) = u_0 \\ u'(0) = u_{00} \end{cases} \quad --- \quad (2)$$

(Note: nth-order ODE needs n ICs).

then (2) is equivalent to having

$$\underset{\sim}{u}(0) = \begin{bmatrix} 0 \\ u_0 \\ u_{00} \end{bmatrix}.$$

§1.2. Linear ODEs.

- Consider the system of 1st-order linear ODEs

$$\underset{\sim}{u}(t) = f(t, \underset{\sim}{u}), \quad \cdots (3)$$

where

$$f(t, \underset{\sim}{u}) = \underset{\sim}{A}(t)\,\underset{\sim}{u} + \underset{\sim}{g}(t).$$

$$\underset{\sim}{u}(t) \in \mathbb{R}^n, \quad \underset{\sim}{A}(t) \in \mathbb{R}^{n\times n}, \quad \underset{\sim}{g}(t) \in \mathbb{R}^n.$$

- If $\underset{\sim}{A}(t) = \underset{\sim}{A}$ is a constant matrix, then (3) is called the <u>constant coeff. linear system</u>.

- If $g(t) = 0$ then it is called the <u>homogeneous</u> system.

- The analytic solution of the homogeneous const. coeff. linear system,

$$\begin{cases} \underset{\sim}{u}' = \underset{\sim}{A}\underset{\sim}{u}, \\ \underset{\sim}{u}(0) = \underset{\sim}{u}_0 \end{cases} \qquad \text{IVP}$$

is $\boxed{\underset{\sim}{u}(t) = \underset{\sim}{u}_0 e^{A(t-t_0)}}$ where the matrix exp. is defined by the Taylor series

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!}(At)^j.$$

- On the other hand the soln of the nonhomogeneous system can be given by

$$\boxed{\underset{\sim}{u}(t) = \underset{\sim}{u}_0 e^{A(t-t_0)} + \int_{t_0}^{t} e^{A(t-s)} g(s)\,ds}$$

This is called the <u>Duhamel's principle</u>.

- Note that the term corresponding to the nonhomog. term $g(s)$ at any instant s has an added effect on the homogeneous soln $\underset{\sim}{u}\, e^{A(t-t_0)}$, given by $e^{A(t-s)} g(s)$.

- We conveniently use $t_0 = 0$.

§1.3. Solution existence and uniqueness.

- In order to guarantee $\exists!$ soln of the IVP:

$$\begin{cases} u'(t) = f(t,u) \\ u(t_0) = u_0 \end{cases} \quad \text{---} \quad (4)$$

it is necessary to require some smoothness of $f(t,u)$.

Thm for Linear ODEs

If $p(t), g(t) \in C(I)$, $I = (\alpha, \beta)$, $t_0 \in I$, then $\exists!$ $u(t) = \phi(t)$, a unique soln of the IVP,

$$\begin{cases} u' + p(t)u = g(t) \\ u(t_0) = u_0. \end{cases}$$

Rmk. If $p(t)$ or $g(t)$ is discontinuous at some point(s), then the soln. can be

(i) discont, or

(ii) non-existent

at those pts.

Thm for nonlinear ODEs.

If f and $\frac{\partial f}{\partial u}$ are continuous on $I = I_t \times I_u$,

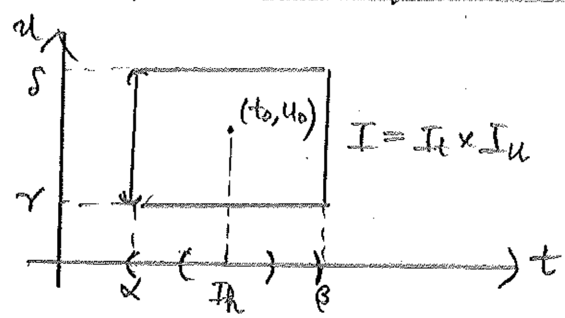
$I_t = (\alpha, \beta)$, $I_u = (\gamma, \delta)$

with $(t_0, u_0) \in I = I_t \times I_u$, then

$\exists$ some interval

$I_h = (t_0 - h, t_0 + h) \subset I_t$, in which

$\exists!$ $u = \phi(t)$, a unique soln of the IVP (4).



Rmk. For linear ODEs, the nonlinear thm reduces to the linear thm

i.e., $u' = f(t,u) = -p(t)u + g(t)$.

Then $\frac{\partial f}{\partial u} = -p(t)$, therefore the continuity of f and $\frac{\partial f}{\partial u}$ is equivalent to the continuity of $p(t)$ and $g(t)$.

Rmk. A stronger version of thm can be also stated using a <u>Lipschitz continuity</u> of f.

Def. $f: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ is Lipschitz cont. (LC) in u over some domain

$$D = [t_0, t_1] \times \Omega$$

if $\exists$ a constant $L \geq 0$ s.t

$$\|f(t,u) - f(t,u^*)\| \leq L \|u - u^*\|,$$

$\forall\ (t,u),\ (t,u^*) \in D.$

Rmk. LC is a stronger form of unif cont (UC), defined by

f is unif. cont if $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t

$$\|u - u^*\| < \delta \Rightarrow \|f(t,u) - f(t,u^*)\| < \varepsilon.$$

Rmk. LC $\Rightarrow \frac{\partial f}{\partial u}$ is bounded (bdd).

Rmk. continuosly differentiable $\subseteq$ LC $\subseteq$ UC.

Ex. If $f \in C^1(D)$, then f is LC,

(pf) $f \in C^1(D) \Rightarrow \frac{\partial f}{\partial u}$ is bdd, let $L = \max_D \left|\frac{\partial f}{\partial u}\right|$.
We now use the mean value thm to find v between u & u^* s.t

$$f(t,u) - f(t,u^*) = \frac{\partial f}{\partial u}(t,v)(u - u^*).$$

$$\Rightarrow \|f(t,u) - f(t,u^*)\| \leq L \|u - u^*\|.$$

Thm. Existence & uniqueness.

If f is LC on D, then $\exists!$ u satisfying

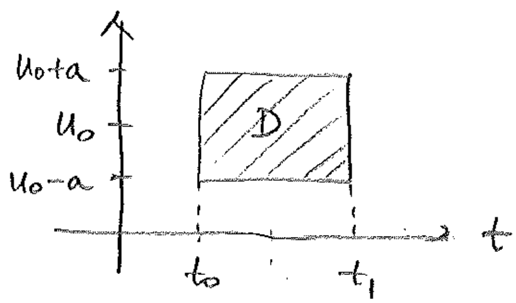
$$\text{IVP} \quad \begin{cases} u'(t) = f(t,u) \\ u(t_0) = u_0 \end{cases}, \qquad D = [t_0, t_1] \times [a, b]$$

for $t < T^*$, where

$$T^* = \min\left(t_1,\ t_0 + \frac{a}{S}\right),$$

$$S = \max_D |f(t,u)| = \max_D |u'(t)|,$$

$$a = \text{radius of } D \text{ centered at } u_0.$$



Ex. $\begin{cases} u'(t) = (u(t))^2 = f(u), \\ u(0) = u_0 > 0. \end{cases}$

Note $f(u) = u^2$ is not LC on $\mathbb{R}$ because $\frac{\partial f}{\partial u} = u$ is not bdd and $\frac{\partial f}{\partial u} \to \infty$ as $u \to \infty$.

But $f(u) = u^2$ is LC on any finite interval $I : |u - u_0| \le a$, because

$$|(u^*)^2 - u^2| \le |u^* + u|\,|u^* - u| \\ \le \underbrace{2(u_0 + a)}_{(=L)}\,|u^* - u|$$

Also, note $\max_I |f(u)| = u'\Big|_{u_0+a} = (u_0+a)^2$.

From the Thm, $\exists!$ soln up to

$$T^* = \frac{a}{(u_0+a)^2}.$$

If we choose a to maximize T^*:

$$0 = \frac{\partial T^*}{\partial a} = \frac{(u_0+a)^2 - 2a(u_0+a)}{(u_0+a)^4} = \frac{u_0^2 - a^2}{(u_0+a)^4}$$

$\Rightarrow a = u_0$

$\Rightarrow \exists!$ soln. up to $T^*\Big|_{a=u_0} = \frac{u_0}{(2u_0)^2} = \frac{1}{4u_0}$.

In fact, the unique analytical soln is

$$u(t) = \frac{1}{\frac{1}{u_0} - t} \longrightarrow \infty \text{ as } t \to \frac{1}{u_0}.$$

Therefore, $\nexists$ no soln. beyond $t = \frac{1}{u_0}$.

Ex. $\begin{cases} u'(t) = \sqrt{u(t)} = f(u) \\ u(0) = 0 = u_0 \end{cases}$

→ Note $f(u) = \sqrt{u}$ is not LC on $\mathbb{R}$, since $\frac{\partial f}{\partial u} = \frac{1}{2\sqrt{u}} \to \infty$ as $u \to 0$.

→ Hence f becomes non LC near $u_0 = 0$,

→ There is no unique soln to this IVP.

→ In fact, $\exists$ two distinct solns

$$\begin{cases} u(t) \equiv 0, & \\ u(t) = \frac{t^2}{4}. \end{cases}$$

→ However, if we modify $u_0 > 0$, then f becomes LC on any finite interval $|u - u_0| \le a$.

Ex. $\begin{cases} u'(t) = \lambda u(t) \\ u(0) = u_0 > 0. \end{cases}$

→ $f(u) = \lambda u$ is LC on $\mathbb{R}$, $\exists!$ soln, given by

$$u(t) = u_0 e^{\lambda t}.$$

→ Case 1 : $\lambda > 0$

The soln diverges as $t \to \infty$.

Case 2 : $\lambda < 0$

The soln converges as $t \to \infty$.

→ Note that the Lipschitz constant λ determines how fast the soln diverges or converges.