Part 2: Challenges In Computing
The Biermann Battery Effects

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From the Last time…

Given the generalized Ohm’s law:

\[
E = -u \times B + \frac{j}{\sigma} + \frac{1}{en_e} j \times B - \frac{1}{en_e} \nabla p_e
\]

For typical large-scale, low-frequency plasma condition, one can approximate:

\[
\frac{OT}{IT} = 10^{-12} << 1, \quad \frac{HT}{IT} = 10^{-1} << 1, \quad \frac{BT}{IT} = 10^{-2} << 1.
\]

(1) ideal MHD: \( E = -u \times B \)

(2) resistive MHD: \( E = -u \times B + \frac{j}{\sigma} \)

(3) Hall MHD: \( E = -u \times B + \frac{1}{en_e} j \times B = -u_e \times B \)

(4) BBT MHD: \( E = -u \times B - \frac{1}{en_e} \nabla p_e \) (Today’s topic!)
Before We Get Going…

take a deep breath.
We are going to see many of these...
It’s Halloween!

Trick-or-Treat!
Math is Sweet!

7 Halloween themed math stations to supplement your teaching theme!

Created by Caitlin O’Bannon @ The Kinder Garden
thekindergarten.blogspot.com
The Biermann Catastrophe

Fatenejad et al., HEDP, 2013

Modeling HEDLA magnetic field generation experiments on laser facilities

Graziani et al., submitted to ApJ

THE BIERMANN CATASTROPHE IN NUMERICAL MHD

Carlo Graziani, Petros Tzeferacos, Dongwook Lee, Donald Q. Lamb, Klaus Weide, Milad Fatenejad, & Joshua Miller
Flash Center for Computational Science, Department of Astronomy & Astrophysics, University of Chicago, Chicago, IL, 60637
Submitted to the Astrophysical Journal

ABSTRACT

The Biermann Battery effect is a popular mechanism for generating magnetic fields in initially unmagnetized plasmas, and is frequently invoked in cosmic magnetogenesis and studied in High-Energy Density laboratory physics experiments. Generation of magnetic fields by the Biermann effect due to mis-aligned density and temperature gradients in smooth flow behind shocks is well known. We show that a magnetic field is also generated within shocks as a result of the electron-ion charge separation that they induce. A straightforward implementation of the Biermann effect in MHD codes does not capture this physical process, and worse, produces unphysical magnetic fields at shocks whose value does not converge with resolution. We show that this breakdown of convergence is due to naive discretization. We show that a careful consideration of the kinetic picture of ion viscous shocks leads to a formulation of the Biermann effect in terms of the electron temperature – which is continuous across shocks – that gives rise to a convergent algorithm. We note two novel physical effects associated with the Biermann effect in shocks: a resistive magnetic precursor in which magnetic field generated by the Biermann effect in the shock “leaks” resistively into the upstream fluid; and a thermal magnetic precursor, in which magnetic field is generated by the Biermann effect ahead of the shock front due to gradients created by the shock’s electron thermal conduction precursor. Both effects appear to be potentially observable in experiments at laser facilities.

Subject headings: magnetohydrodynamics — plasmas — magnetic fields
To Investigate B-field in the Universe

- In the universe, shocks are driven when two or more giant galaxy clusters are merging together by gravitational collapsing.

- Mass accretion onto these clusters generate high Mach number shocks.

- These shocks can form a tiny “seed” magnetic fields which can then be amplified by turbulent dynamo processes.

- FRM inferred $B \sim 1.2 \, \mu$G.

Courtesy of F. Miniati (ETH)

Courtesy of A. Kravtsov
U of Chicago
Seed Magnetic Fields

• Generation of such seed magnetic fields is interesting and important in understanding dynamo theories.

• Magnetic field generation is also important for experiments at laser facilities such as NIF (National Ignition Facility @ LLNL).

• Understanding the mechanism of a seed magnetic field is crucial for both astrophysics and high-energy-density physics (HEDP).

• What do we know already? Where should we start?
  • The famous Biermann Battery Mechanism (Biermann, 1950)
• The resistive induction equation,

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left\{ \mathbf{u} \times \mathbf{B} - \frac{\eta c^2}{4\pi} \nabla \times \mathbf{B} \right\}
\]

has the solution \( \mathbf{B}(\mathbf{x},t) = 0 \), and therefore cannot generate magnetic field ab initio.
Resistive MHD + BBT

• The resistive induction equation,

\[ \frac{\partial B}{\partial t} = \nabla \times \left\{ u \times B - \frac{\eta c^2}{4\pi} \nabla \times B \right\} \]

has the solution \( B(x,t) = 0 \), and therefore cannot generate magnetic field ab initio.

• The Biermann Battery Term (BBT) provides a work-around and is a popular resolution of generating a seed magnetic field.

• Thermal charge separation results in an electric field

\[ \mathbf{E}_B \equiv -\left( e n_e \right)^{-1} \nabla P_e, \]

that can jump-start magneto-genesis in non-barotropic flows.
Resistive MHD + BBT

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\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left\{ \mathbf{u} \times \mathbf{B} - \frac{\eta c^2}{4\pi} \nabla \times \mathbf{B} \right\} - c \nabla \times \mathbf{E}_B
\]

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• The Biermann Battery Term (BBT) provides a work-around and is a popular resolution of generating a seed magnetic field.

• Thermal charge separation results in an electric field

\[
\mathbf{E}_B \equiv -(en_e)^{-1} \nabla P_e,
\]

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Resistive MHD + BBT

• The resistive induction equation,

\[
\frac{\partial B}{\partial t} = \nabla \times \left\{ \mathbf{u} \times \mathbf{B} - \frac{\eta c^2}{4\pi} \nabla \times \mathbf{B} \right\} - \frac{c}{en_e^2} \nabla n_e \times \nabla P_e
\]

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BBT at Shocks

\[ \frac{\partial B}{\partial t} \bigg|_{\text{BBT}} = \frac{c \nabla P_e \times \nabla n_e}{q_e n_e^2} \]

- BBT generates B fields when gradients of electron pressure and density are not aligned.
- BBT is zero in 1D (or symmetric) flow or at spherical shocks.
- BBT becomes non-zero when symmetry is broken, and two gradients are not aligned to each other.
- Shocks should furnish ideal sites for Biermann-effect magneto-genesis.

In HEDP experiment, laser is used to drive asymmetric shocks!
But there’s a problem...

- Numerical implementations of BBT in MHD codes work fine in smooth flow. However, they fail catastrophically as soon as shocks develop.

- The problem is the Biermann flux

\[ \mathbf{E}_B \equiv -\left( e n_e \right)^{-1} \nabla P_e, \]

at discontinuities. The derivative increases without bound with increasing resolution.
Failure in Grid Convergence

- Magnetic energy, $E_{\text{mag}}$, fails to converge with grid resolution in both approaches of a naive flux formulation and a source term.

- This type of BBT catastrophe at shocks has been an issue for long time in many existing MHD codes in both Eulerian (e.g., CRASH, ENZO, RAGE/CASSIO) and ALE (e.g., HYDRA) based formulations.
One Attempted Effort

- Suppress the BB generation across the shock by utilizing a shock detection algorithm.

- Although not correct, we’ve seen a convergence behavior of the magnetic energy with grid resolution (Fatenejad et al. 2013)
What Went Wrong???

• The Biermann “flux” of magnetic field is proportional to the gradient of electron pressure. It is well-defined at smooth flows, while ill-defined at shocks where the electron pressure becomes discontinuous.

• This is a serious embarrassment: the Biermann effect has been invoked for 20+ years precisely at shocks, since that’s where the effect is expected to be largest.

• But that’s precisely where the theory breaks down (and nobody has noticed!).
Why Did It Go Wrong???

• In order to help understanding the issue with:

\[ \nabla \times \mathbf{E}_B = \nabla \times \left( \frac{\nabla P_e}{e n_e} \right) = -\frac{1}{e n_e^2} \nabla n_e \times \nabla P_e \]

• Take two scalar functions \( f \) & \( g \) whose gradients are collinear:

\[ \nabla g = \alpha \nabla f \]

• They satisfy

\[ \nabla g \times \nabla f = 0 \]
Why Did It Go Wrong???

• In order to help understanding the issue with:

\[ \nabla \times \mathbf{E}_B = \nabla \times \frac{\nabla P_e}{en_e} = - \frac{1}{en_e^2} \nabla n_e \times \nabla P_e \]

• Taylor expand both to third order, and take averages in FV sense, and taking the “discrete curl” of \( g \nabla f \) gives:

\[ \nabla \times g \nabla f = CO(\Delta^2) \]

where \( C \) is homogeneous of order 3 in the derivatives of \( f \) and \( g \).

• \( \nabla \times g \nabla f = O(\Delta^2) \) in smooth flow;

\[ \nabla \times g \nabla f = O(\Delta^{-1}) \] in discontinuouos flow (divergent!)
• In a plasma, the electrons have much higher mobility - and hence thermal conductivity - than ions, due to the high mass ratio $m_i/m_e$.

• This means that the electron temperature is continuous, even at shocks, where $n_e, P_e$ are discontinuous.

Temperature structure of plasma shock. Zel'dovich & Rizer, Physics of shock waves and high-temperature hydrodynamics phenomena.
Salvation in Kinetic Theory

- Two-temperature treatment of plasma is essential.
- Ions are dissipative; electrons are adiabatic.
- An electron thermal conduction rules out any sudden change in $T_e$ at the shock.
- An electron conduction precursor leads the shock.

Temperature structure of plasma shock calculated with FLASH.
Salvation in Te Continuity

- We can leverage Te continuity by re-writing the BBT:

\[ n_e^{-1} \nabla P_e = k_B T_e \nabla \ln P_e \]

- Hence we get

\[ \frac{\partial B}{\partial t} \bigg|_B = \frac{c k_B}{e} \nabla T_e \times \nabla \ln P_e \]

- Compare this to the original BBT:

\[ \frac{\partial B}{\partial t} \bigg|_B = -\frac{c}{en_e^2} \nabla n_e \times \nabla P_e \]
Salvation in Te Continuity

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• Hence we get

\[ \frac{\partial B}{\partial t} \bigg|_B = \frac{ck_B}{e} \nabla T_e \times \nabla \ln P_e \]

• Better, but done not yet, because

\[ \nabla T_e, \ln P_e \]

are discontinuous at the shock.
Shock Surface

• Define the shock surface \( \Gamma = \{ \mathbf{x} | \Psi(\mathbf{x}, t) = 0 \} \)

\[
\frac{\partial \Psi}{\partial t} + D |\nabla \Psi| = 0
\]

• Use this to represent any discontinuous profiles, i.e., \( \nabla T_e, \ln P_e : \)

\[
\nabla T_e = \Theta(\Psi)T_{eu} + \Theta(-\Psi)T_{ed}
\]

\[
\ln P_e = \ln P_{eu} \Theta(\Psi) + \ln P_{ed} \Theta(-\Psi)
\]
Finite Biermann Flux

• Taking derivative of \( \ln P_e = \ln P_{eu} \Theta(\Psi) + \ln P_{ed} \Theta(-\Psi) \) using \( \frac{d\Theta(\Psi)}{d\Psi} = \delta(\Psi) \) gives:

\[
\nabla \ln P_e = \ln \left( \frac{P_{eu}}{P_{ed}} \right) \delta(\Psi) \nabla \Psi + \Theta(\Psi) \nabla \ln P_{eu} + \Theta(-\Psi) \nabla \ln P_{ed}
\]

• One obtains a “finite” jump in B at shocks:

\[
\Delta B \propto \ln \left( \frac{P_{eu}}{P_{ed}} \right) \nabla T_e \times \mathbf{n}, \text{ where } \mathbf{n} = \frac{\nabla \Psi}{|\nabla \Psi|}
\]

• Lastly, the correctly behaving Biermann flux can be obtained after considering the Rankine-Hugoniot relation for B:

\[
f_B(\mathbf{n}) = -\frac{ck_B}{e} (\ln P_e) \nabla T_e \times \mathbf{n}
\]
Taming the Biermann Catastrophe

- Almost ready now… we also need the electron entropy advection:

\[
\frac{\partial \rho s_e}{\partial t} + \nabla \cdot (\rho u s_e) = -T_e^{-1} \nabla \cdot (-\kappa_e \nabla T_e) + \frac{\rho c_{v,e}}{T_e \tau_{ei}} (T_i - T_e)
\]

- And, don’t forget there’s a Biermann term in the energy flux too!

\[
f_{\rho E}(n) = \frac{ck_B}{4\pi e} \ln P_e \nabla \times (T_e B)
\]
To summarize, at a cell interface with a normal vector \( n \), the hydro fluxes are to be adapted to the Biermann effect by adding an additional flux vector given by

\[
F^{(B)} = \begin{pmatrix}
F^{(B)}_\rho \\
F^{(B)}_{\rho u} \\
F^{(B)}_\rho \varepsilon \\
F^{(B)}_B
\end{pmatrix}
\equiv
\begin{pmatrix}
0 \\
0 \\
f_{\rho \varepsilon}(n) \\
f_B(n)
\end{pmatrix}
\]

\[
f_B(n) = - \frac{c k_B}{e} (\ln P_e) \nabla T_e \times n
\]

\[
f_{\rho E}(n) = \frac{c k_B}{4 \pi e} \ln P_e \nabla \times (T_e \mathbf{B})
\]
• FLASH simulation of a Sedov-like *spherical* explosion, 2D cylindrical symmetry
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Verification: Ellipsoidal Shock

- FLASH simulation of a Sedov-like *ellipsoidal* explosion, 2D cylindrical symmetry
Verification: Ellipsoidal Shock

- Define the shock condition parameter \( C \):

\[
C \equiv \text{Norm.} \times \left\{ (D - u_d)B_d - (D - u_u)B_u + \frac{ck_B}{e} \left( n_z \frac{\partial T_e}{\partial R} - n_R \frac{\partial T_e}{\partial z} \right) \left[ \ln P_{e,d} - \ln P_{e,u} \right] \right\}
\]

- \( C \) should always be zero up to some truncation or rounding precision.
• Recall the physical basis for the **electron conduction shock precursor**: a balance of heat diffusion out of the shock and heat advection into the shock, in the presence of impulsive heating at the shock.

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**Fig. 7.20.** Temperature and density distributions for a strong shock wave in a plasma (the figure is taken from [43]). The electron temperature at the compression shock $T_{e0} = 0.93 T_1$; the ion temperatures ahead and behind the compression shock are $T_{i1} = 0.16$ and $T_{i2} = 1.24 T_1$, respectively. The densities ahead and behind the compression shock $\rho_{1}/\rho_0 = 1.13$ and $\rho_{2}/\rho_0 = 3.53$. 
A similar effect occurs for B in the presence of finite resistivity: the balance of magnetic diffusion out of the shock with magnetic advection into the shock in the presence of impulsive magnetogeneration by BBT at the shock gives rise to a resistive magnetic shock precursor.

The precursor is exponential, with a skin depth

\[ \lambda_B = \frac{c^2 \eta}{4\pi D} \]
• Assuming Spitzer-type diffusion, $\lambda_B$ scales as follows:

$$\lambda_B = 16.4 \text{ cm} \times Z \times \frac{\ln \Lambda}{10} \times \left( \frac{D}{10^6 \text{ cm s}^{-1}} \right)^{-1} \times \left( \frac{k_B T_e}{1 \text{ eV}} \right)^{-3/2}$$
Conclusions

• The Biermann Catastrophe is a nasty numerical pathology that afflicts naive numerical implementations of BBT in the presence of shocks.

• It is curable by importing elements from the kinetic theory of plasma shocks, in particular by exploiting the continuity of electron temperature at shocks.

• A new physical effect, the resistive magnetic shock precursor, is potentially observable in HEDP experiments.