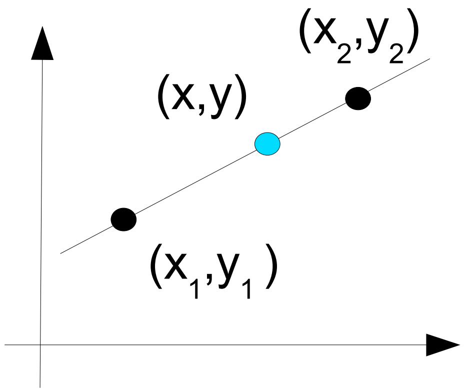
Review Objectives

- To review different forms of equations of lines and how to write the specific equation given two points, point and the slope, etc;
- To review systems of linear equations with both two and three variables by using the technique of elimination by addition or by substitution;
- To review quadratic functions;
- To review exponential and logarithmic functions;
- To review techniques for solving logarithmic and exponential equations

Lines

Slope *m is the ratio between vertical and horizontal change*

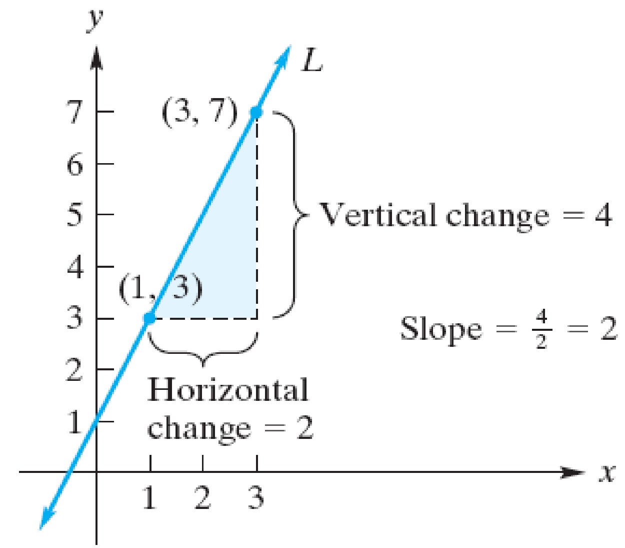


$$\frac{y_2 - y_1}{x_2 - x_1} = m$$

For any point (x,y) on the line, the slope is same; therefore,

$$\frac{y - y_1}{x - x_1} = m$$

Example



This is an implicit relation between x and y; let us find the explicit relation (find y as a function of x)

$$\frac{y - y_1}{x - x_1} = m \Rightarrow y - y_1 = m(x - x_1) \quad (*)$$

or $y = m(x - x_1) + y_1$

$$y = mx - mx_1 + y_1$$

If we introduce $b = y_1 - mx_1$

$$y = mx + b \quad (**)$$

The form $(*)$ is the point-slope form: $y - y1 = m(x - x_1)$

The form $(**)$ is the slope-intercept form: $y = mx + b$

In general, any line can be presented in the general form as $Ax + By + C = 0$

If the line is 5x+3y+4=0, can you find the slope-intercept form of it?

(Hint: express y as a function of x)

Lines

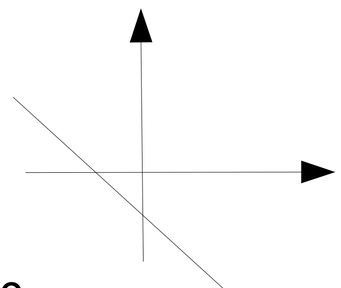
If the line is 5x+3y+4=0, can you find the slope intercept form of it?

$$5x+3y+4=0$$

$$3y=-5x-4$$

$$y=-\frac{5}{3}x-\frac{4}{3}$$

The slope is negative



At which point does this line cross y axis?

(Hint: when the line crosses y, we know that x=0)

Approach 1: Use the original form of the line and find y for x=0

$$5x+3y+4=0 \Rightarrow \quad 5\cdot 0+3y+4=0$$

$$3y+4=0$$

$$3y=-4$$

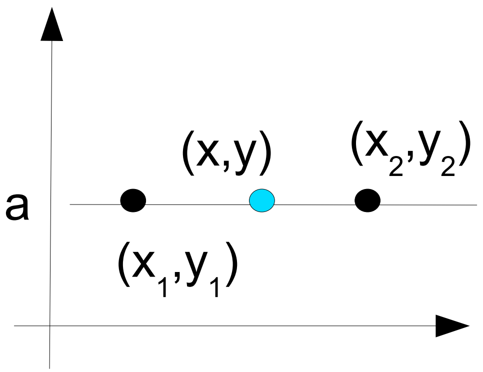
$$y=-\frac{4}{3}$$

Approach 2: Use the slope-intercept form

$$y=-\frac{5}{3}x-\frac{4}{3} \Rightarrow y=-\frac{5}{3}\cdot 0-\frac{4}{3}=-\frac{4}{3}$$

The point is x=0, y = -4/3 , or (0, -4/3)

Lines



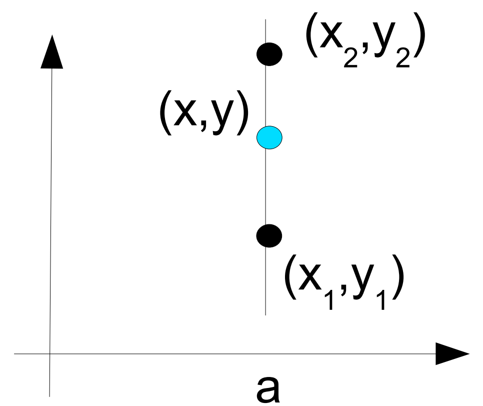
Horizontal line y=a

$$\frac{y_2 - y_1}{x_2 - x_1} = m$$

$$m = 0$$

$$\frac{y - y_1}{x - x_1} = 0 \Rightarrow y = y_1 \Rightarrow y = a$$

y does not depend on x



Vertical line x=a

$$\frac{y_2 - y_1}{x_2 - x_1} = m$$

$$\frac{y_2 - y_1}{x_1 - x_1} = \frac{y_2 - y_1}{0}$$

we can say that the slope is +"infinite", or -"infinite"

Lines summary

The point-slope form $y - y1 = m(x - x_1)$ (Example $y - 5 = 0.5(x - 2)$)

The point-intercept form $y = mx + b$ (Example $y = \frac{2}{3}x + 2$)

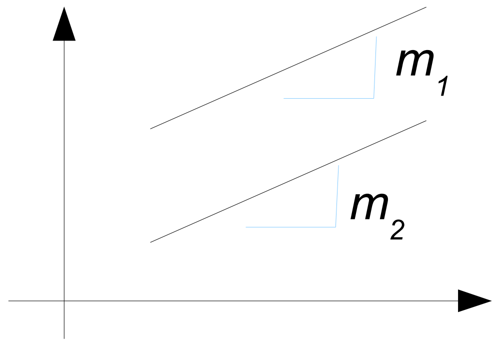
The general form $Ax + By + C = 0$ (Example $2x + \frac{2}{3}y + \frac{1}{4} = 0$)

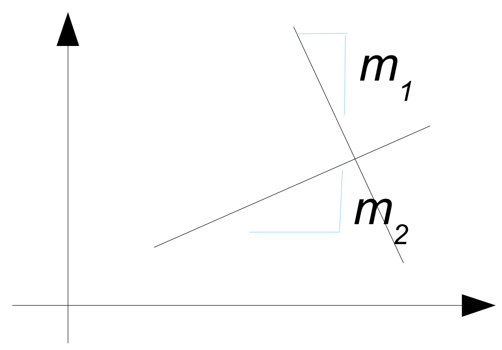
Horizontal line $y = a$ (Example $y = \frac{7}{5}$)

Vertical line $x = a$ (Example $x = 2.3$)

Two lines $y = m_1 x + b_1$ and $y = m_2 x + b_2$

- are parallel if they have the same slope ($m_1 = m_2$).
- are mutually normal if $m_1 = -\frac{1}{m_2}$





Special case:
Horizontal line (m=0), and
vertical line (m=-"infinite")

Problems

1. Find the line which has slope 2 and passes through (-1,3).

2. What are the points where the line (1) crosses x and y axis?

3. Find where the line 3x+5y+4=7 crosses x and y axis.

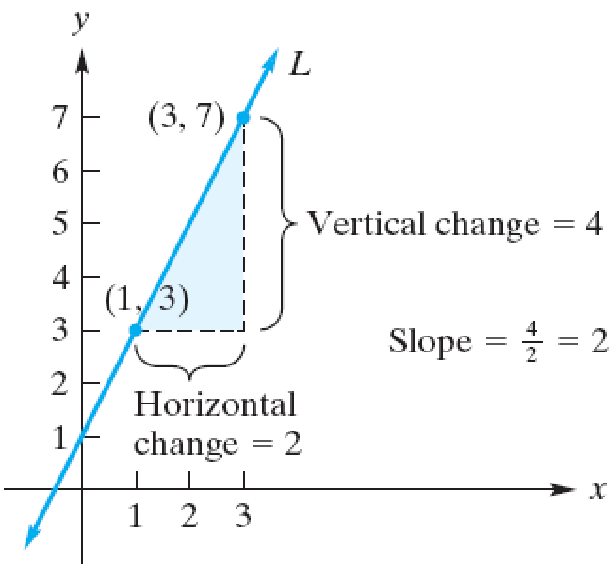
4. Find the slope of the line (3).

5. Find the line which is normal to the line y=3 and goes through the point (2,2).

6. Find the line which is parallel to the line y=3 and goes through the point (5,4).

7. Find the intersection of these two lines $x+2y-8=0$ and $3x+4y+4=0$

8. Write the slope-intercept form for the line in the figure



Systems of Linear Equations

Elimination by substitution

$$x+2y-8=0$$
$$3x+4y+4=0$$

$$x=-2y+8$$
$$3x+4y+4=0$$

$$x=-2y+8$$
$$3(-2y+8)+4y+4=0$$

$$x=-2y+8$$
$$-6y+24+4y+4=0$$

$$x=-2y+8$$
$$-2y=-28$$

$$x=-2y+8$$
$$y=14$$

$$x=-28+8=-20$$
$$y=14$$

The solution is x=-20, y=14

Elimination by substitution

$$x+2y-8=0$$
$$2x+4y+4=0$$

$$x=-2y+8$$
$$2x+4y+4=0$$

$$x=-2y+8$$
$$2(-2y+8)+4y+4=0$$

$$x=-2y+8$$
$$-4y+16+4y+4=0$$

$$x=-2y+8$$
$$20=0$$

There is no solution. Why ?
(Hint: check the slopes)

$$x+2y-8=0 \Leftrightarrow y=-\frac{1}{2}x+8$$
$$2x+4y+4=0 \Leftrightarrow y=-\frac{1}{2}x-2$$

Lines have the same slope (they are parallel); therefore, there is no point that belongs to both of them.

Elimination by addition

$$x+2y-8=0$$
$$3x+4y+4=0$$

$$-3x-6y+24=0$$
$$3x+4y+4=0$$

$$-2y+28=0 \Leftrightarrow y=14$$
$$x+2\cdot 14-8=0 \Leftrightarrow x=-20$$

Systems of Linear Equations

Infinitely many solutions

$$x+5y=2$$
$$\frac{1}{2}x+\frac{5}{2}y=1$$

$$x+5y=2$$
$$x+5y=2$$

The same equation

$$x+5y=2$$

$$y=r$$
$$x=-5r+2$$

Three variables

$$2x+y+z=3$$
$$-x+2y+2z=1$$
$$x-y-3z=-6$$

$5y+5z=5$ From Eq.(1)+2Eq.(2)

$y-z=-5$ From Eq.(2)+Eq.(3)

$$y+z=1$$
$$y-z=-5$$

$$2y=-4\Rightarrow y=-2$$
$$\Rightarrow z=3$$

$$x-y-3z=-6$$
$$x-(-2)-9=-6\Rightarrow x=7-6=1$$

Book page 154, solution by the substitution method

In the case of three variables, we can have a solution, no solution, or infinitely many solutions. In the case of infinitely many solutions, we may need one, or two free variables

$$x-2y-z=0$$
$$2x-4y-2z=0$$
$$-x+2y+z=0$$

Eq.(1), Eq(2) and Eq(3) are equivalent to Eq(1).

$$x-2y-z=0$$

The solution is x = 2y+z, y=r, z=s ,i.e., x = 2r+s, y=r, z=s

Quadratic function

A function *f* is quadratic if and only if *f(x)* can be written in the form $f(x)=ax^2+bx+c$, where *a*, *b* and *c* are constants and $a\neq 0$

Note: If *a*=0 $f(x)=0\cdot x^2+bx+c$

$f(x)=bx+c$, which is line

Examples:

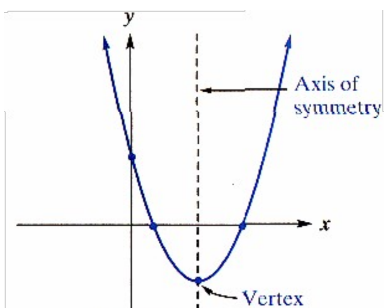
$y(x)=x^2-3x+2$, or $y=x^2-3x+2$

$F(t)=-3t^2$

$f(s)=c(s-a)(s-b)=c(s^2-(a+b)s+ab)=cs^2-c(a+b)s+cab \quad \text{if } c\neq 0$

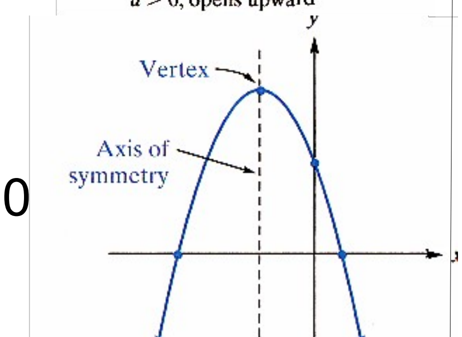
$f(x)=ax^2+bx+c$

$a>0$



$a>0$, opens upward

$a<0$



$a<0$, opens downward

Find the point at which the quadratic function $f(x)=ax^2+bx+c$ crosses y axis.

When the line crosses y, we know that x=0

$$f(0)=a\cdot 0^2+b\cdot 0+c=c$$

The point is $(0,c)$

Example: If $y=2x^2+5x+3$

at the point at which the quadratic function crosses y axis, x=0

$y=2\cdot 0^2+5\cdot 0+3=3$

The point at which the quadratic function crosses y axis is (0,3)

Quadratic function

Find points at which $f(x)=ax^2+bx+c=0$

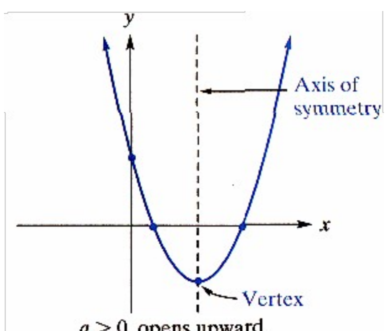
i.e., the points where the quadratic function crosses x axis

In the figures, there are two points f(x)=0 and we can compute them as

$$x_1=\frac{-b-\sqrt{b^2-4ac}}{2a}, \qquad x_2=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

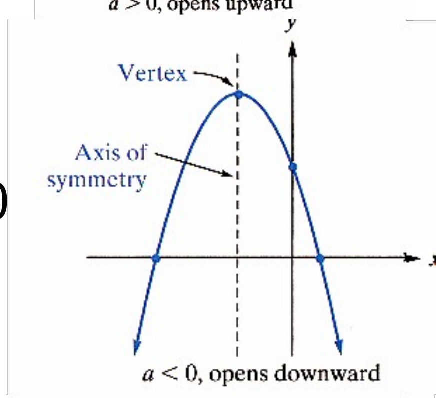
$f(x)=ax^2+bx+c$

$a>0$



$a>0$, opens upward

$a<0$



$a<0$, opens downward

If $b^2-4ac<0$, then the quadratic function does not cross x axis, i.e., there is no point f(x)=0

It is because the square root of negative value is undefined

If $b^2-4ac=0$, then $x_1=x_2=-\frac{b}{2a}$

In any case, the x position of the vertex is $x_v=-\frac{b}{2a}$

which is formally $x_v=\frac{x_1+x_2}{2}$

Q: What is $f(x_v)$? A: $f(x_v)=ax_v^2+bx_v+c$

The vertex is at $(x_v, f(x_v))$

Quadratic function

Very important: If $b^2 - 4ac \geq 0$, every expression $ax^2 + bx + c$

can be represented as $a(x - x_1)(x - x_2)$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

Simplify $\dfrac{2x^2 - 10x + 12}{x^2 - 11x + 24}$ assuming $x^2 - 11x + 24 \neq 0$

$2x^2 - 10x + 12$: $b^2 - 4ac = 100 - 4 \cdot 2 \cdot 12 = 100 - 96 = 4$

$$x_{1,2} = \frac{10 \pm \sqrt{4}}{2 \cdot 2} = \frac{10 \pm 2}{4} \Rightarrow x_1 = 12/4 = 3, x_2 = 8/4 = 2$$

$$2x^2 - 10x + 12 = 2(x - 3)(x - 2)$$

$x^2 - 11x + 24$: $b^2 - 4ac = 121 - 4 \cdot 1 \cdot 24 = 121 - 96 = 25$

$$x_{1,2} = \frac{11 \pm \sqrt{25}}{2 \cdot 1} = \frac{11 \pm 5}{2} \Rightarrow x_1 = 16/2 = 8, x_2 = 6/2 = 3$$

$$x^2 - 11x + 24 = (x - 8)(x - 3)$$

$$\frac{2x^2 - 10x + 12}{3x^2 - 18x + 24} = \frac{2(x - 3)(x - 2)}{(x - 8)(x - 3)} = \frac{2(x - 2)}{x - 8}$$

System of Linear and Quadratic Equations

Solve:

$$x^2 - 2x + y - 7 = 0$$
$$3x - y + 1 = 0$$

$$y = 3x + 1$$

$$x^2 - 2x + 3x + 1 - 7 = 0$$

$$x^2 + x - 6 = 0$$

$$x_{12} = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} \Rightarrow x_1 = 2, x_2 = -3$$

Consequently, y has two solutions:

$$y_1 = 3x_1 + 1 = 7$$
$$y_2 = 3x_2 + 1 = -8$$

There are two solutions, (2,7) and (-3,-8).

Always remember:

$$(a+b)^2=a^2+2ab+b^2$$

$$(a-b)^2=a^2-2ab+b^2$$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3$$

$$a^2-b^2=(a-b)(a+b)$$

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

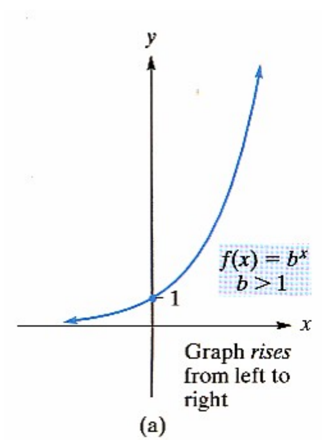
Exponential Functions

The function *f* defined by $f(x)=b^x$, where $b>0, b\neq 1$, and the exponent *x* is any real number, is called an **exponential function** with the base *b*.

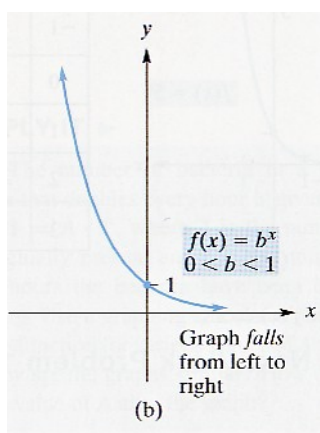
Properties:

1. $b^x b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(bc)^x = b^x c^x$
5. $\left(\dfrac{b}{c}\right)^x = \dfrac{b^x}{c^x}$
6. $b^{-x} = \dfrac{1}{b^x}$
7. $b^1 = b$
8. $b^0 = 1$
9. $b^{\frac{x}{y}} = \sqrt[y]{b^x} = (\sqrt[y]{b})^x$

$f(x)=b^x, b>1$



$f(x)=b^x, 0<b<1$



1. The domain of any exponential function is $(-\infty,\infty)$
   The range of any exponential function is $(0,\infty)$
2. The graph has *y*-intercept (0,1).
   There is no *x*-intercept.
3. If b>1, the graph rises from left to right.
   If 0<b<1, the graph falls from left to right.
4. If b>1, the graph approaches 0 for negative x.
   If 0<b<1, the graph approaches 0 for positive x.

Examples:

$$2^{-\frac{3}{2}}=\frac{1}{2^{\frac{3}{2}}}=\frac{1}{\sqrt[2]{2^3}}=\frac{1}{\sqrt[2]{8}}=\frac{1}{\sqrt[2]{2\cdot 4}}=\frac{1}{\sqrt[2]{2}\cdot\sqrt[2]{4}}=\frac{1}{2\sqrt{2}}$$

$$=\frac{1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{4}$$

$$\frac{1}{\sqrt{3}-\sqrt{2}}=\frac{1}{\sqrt{3}-\sqrt{2}}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{\sqrt{3}+\sqrt{2}}{3-2}=\frac{\sqrt{3}+\sqrt{2}}{1}=\sqrt{3}+\sqrt{2}$$

$$\frac{3^{\frac{3}{8}}}{9^{\frac{2}{3}}}=\frac{3^{\frac{3}{8}}}{(3^2)^{\frac{2}{3}}}=\frac{3^{\frac{3}{8}}}{3^{2\cdot\frac{2}{3}}}=\frac{3^{\frac{3}{8}}}{3^{\frac{4}{3}}}=3^{\frac{3}{8}-\frac{4}{3}}=3^{\frac{3\cdot 3}{24}-\frac{4\cdot 8}{24}}=3^{\frac{9-32}{24}}=3^{-\frac{21}{24}}=3^{-\frac{7}{8}}$$

$$=\frac{1}{3^{\frac{7}{8}}}=\frac{1}{\sqrt[8]{3^7}}$$

Logarithmic Functions

The functions inverse to the exponential functions are called the logarithmic functions.

$$y = \log_b x \quad \text{if and only if} \quad b^y = x$$

$$1 < b \qquad \text{or} \qquad 0 < b < 1$$

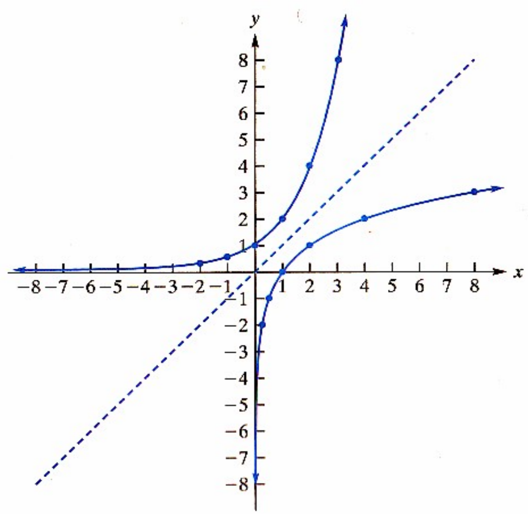


FIGURE 4.16 Graphs of $y = 2^x$ and $y = \log_2 x$.

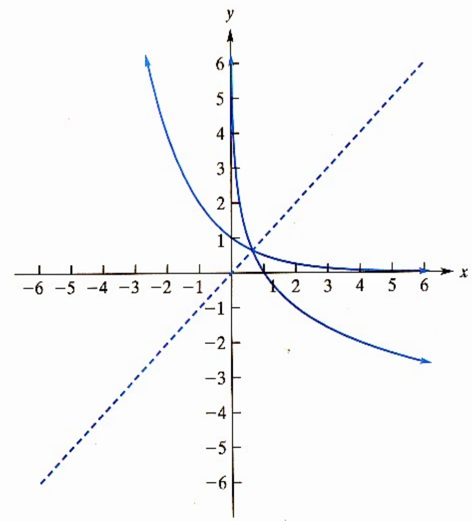


FIGURE 4.18 Graph of $y = (\frac{1}{2})^x$ and $y = \log_{1/2} x$.

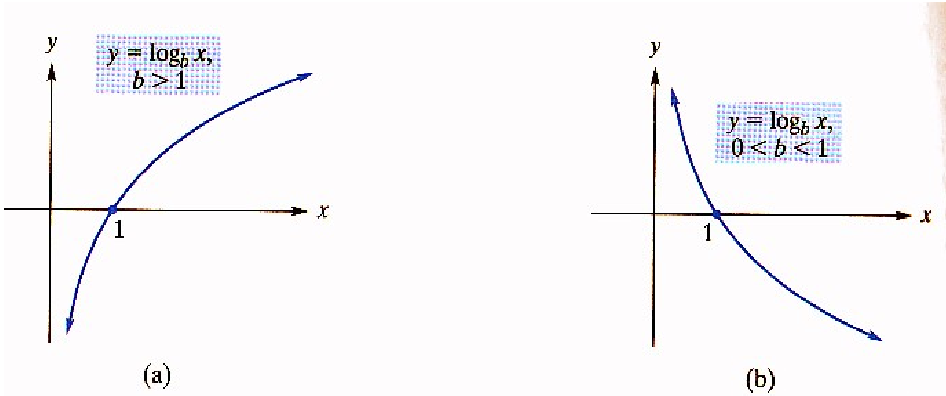


FIGURE 4.19 General shapes of $y = \log_b x$.

1. The domain is $(0, \infty)$
   The range is $(-\infty, \infty)$
2. The graph has x-intercept (1,0).
   There is no y-intercept
3. If b>1, the graph rises from left to right.
   If 0<b<1, the graph falls from left to right.

Property of logarithms

$$1.\ \log_b(mn)=\log_b m+\log_b n \qquad 4.\ \log_b\left(\frac{1}{m}\right)=-\log_b m \qquad 7.\ \log_b b^r=r$$

$$2.\ \log_b\left(\frac{m}{n}\right)=\log_b m-\log_b n \qquad 5.\ \log_b 1=0 \qquad 8.\ b^{\log_b m}=m$$

$$3.\ \log_b m^r=r\log_b m \qquad 6.\ \log_b b=1 \qquad 9.\ \log_b m=\frac{\log_a m}{\log_a b}$$

$$\log_{10} x=\log x$$

$$\log_e x=\ln x \qquad e=2.71828182845\ldots$$

Examples:

Find $\log_{36} 6$?

$$\log_{36} 6=\frac{\log 6}{\log 36}=\frac{\log 6}{\log 6^2}=\frac{\log 6}{2\log 6}=\frac{1}{2}$$

Find x in: $\log_x(2x-3)=1 \Rightarrow (2x-3)=x^1 \Rightarrow x=3$

Find x in: $\log_x(16-4x-x^2)=2 \Rightarrow 16-4x-x^2=x^2 \Rightarrow 2x^2+4x-16=0$

$$x^2+2x-8=0 \Rightarrow \quad x_1=\frac{-2-\sqrt{36}}{2}=-4$$

$$x_2=\frac{-2+\sqrt{36}}{2}=2$$

x_1 cannot be the solution because it is the logarithm basis, and the basis must be positive

Examples:

Find x in: $\log_2 x = 5 - \log_2(x+4)$

$$\log_2 x + \log_2(x+4) = 5$$

$$\log_2 x(x+4) = 5 \Rightarrow \quad x^2 + 4x - 32 = 0 \Rightarrow \quad \begin{aligned} x_1 &= \frac{-4+\sqrt{16+128}}{2} = \frac{-4+\sqrt{144}}{2} = 4 \\ x_2 &= \frac{-4-\sqrt{16+128}}{2} = \frac{-4-\sqrt{144}}{2} = -8 \end{aligned}$$

x_2 cannot be the solution because on the right side of the original equation *x* must be positive

If $p = 12^{1-0.1q}$, express q as a function of *p*

$$\log p = (1-0.1q)\,log12$$

$$\frac{\log p}{\log 12} = 1 - 0.1q$$

$$0.1q = 1 - \frac{\log p}{\log 12}$$

$$q = 10\left(1 - \frac{\log p}{\log 12}\right)$$

Radioactive decay half-time (*T*)

$$N(t) = N_0 e^{-\lambda t}$$

$$\frac{N_0}{2} = N_0 e^{-\lambda T}$$

$$\frac{1}{2} = e^{-\lambda T}$$

$$-\ln 2 = -\lambda T$$

$$T = \frac{\ln 2}{\lambda}$$

Radioactive decay

N_0 the amount of the element present at t=0

λ the decay constant

$N(t)$ the amount of the element present at t

$$N(t)=N_0 e^{-\lambda t}$$

$$\frac{N_0}{2}=N_0 e^{-\lambda T}$$

T is the half-life time and by this definition the following applies

$$N_0 e^{-\lambda(t+T)}=N_0 e^{-\lambda T} e^{-\lambda t}=\frac{N_0}{2} e^{-\lambda t}=\frac{1}{2} N(t)$$

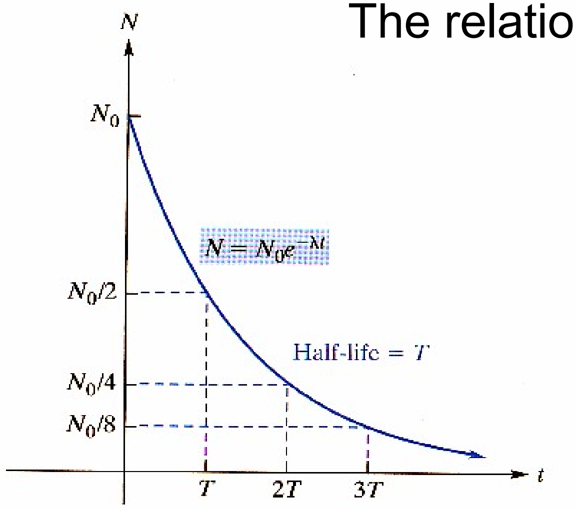


FIGURE 4.13 Radioactive decay.

The relation between T and λ can be derived as follows:

$$\frac{N_0}{2}=N_0 e^{-\lambda T}$$

$$\frac{1}{2}=e^{-\lambda T}$$

$$-\ln 2=-\lambda T$$

$$T=\frac{\ln 2}{\lambda}$$

Transformations of Exponential Functions

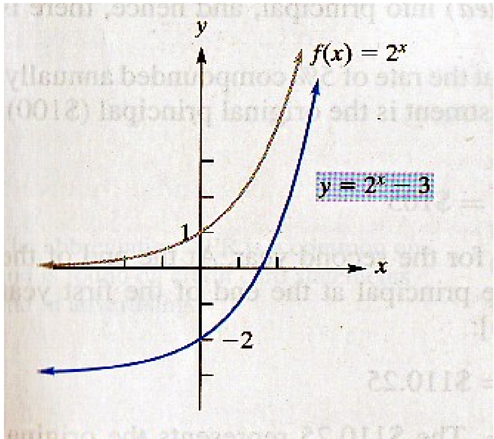


FIGURE 4.4 Graph of $y = 2^x - 3$.

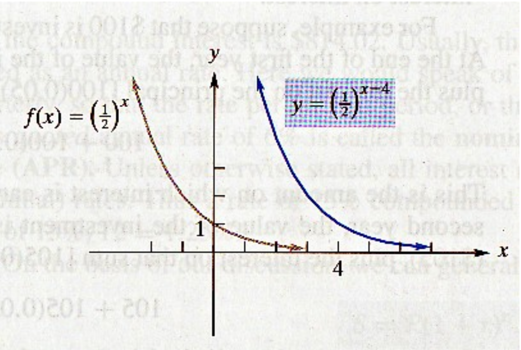
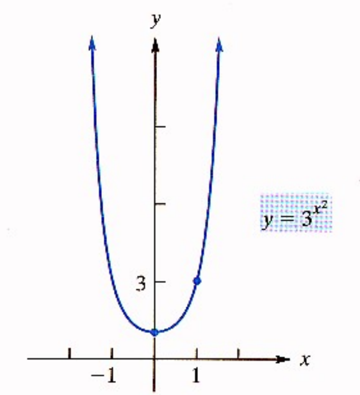


FIGURE 4.5 Graph of $y = \left(\frac{1}{2}\right)^{x-4}$.



x	0	1	2
y	1	3	81

FIGURE 4.6 Graph of $y = 3^{x^2}$.

Always have in mind the following rules + algebra:

$(a+b)^2=a^2+2ab+b^2$

$(a-b)^2=a^2-2ab+b^2$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$(a-b)^3=a^3-3a^2b+3ab^2-b^3$

$a^2-b^2=(a-b)(a+b)$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$a^3+b^3=(a+b)(a^2-ab+b^2)$

1. $b^x b^y=b^{x+y}$
2. $\dfrac{b^x}{b^y}=b^{x-y}$
3. $(b^x)^y=b^{xy}$
4. $(bc)^x=b^x c^x$
5. $\left(\dfrac{b}{c}\right)^x=\dfrac{b^x}{c^x}$
6. $b^{-x}=\dfrac{1}{b^x}$
7. $b^1=b$
8. $b^0=1$
9. $b^{\frac{x}{y}}=\sqrt[y]{b^x}=(\sqrt[y]{b})^x$

1. $\log_b(mn)=\log_b m+\log_b n$
2. $\log_b\left(\dfrac{m}{n}\right)=\log_b m-\log_b n$
3. $\log_b m^r=r\log_b m$
4. $\log_b\left(\dfrac{1}{m}\right)=-\log_b m$
5. $\log_b 1=0$
6. $\log_b b=1$
7. $\log_b b^r=r$
8. $b^{\log_b m}=m$
9. $\log_b m=\dfrac{\log_a m}{\log_a b}$

$\log_{10} x=\log x$

$\log_e x=\ln x \qquad e=2.71828182845...$

Example:

$$\log_{\sqrt{2}} 8 = \frac{\ln 8}{\ln \sqrt{2}} = \frac{3 \ln 2}{\frac{1}{2} \ln 2} = \frac{3}{\frac{1}{2}} = 6$$

Obviously based on the logarithm definition

$$(\sqrt{2})^6 = 8$$

$$\text{check} (\sqrt{2})^6 = 2^{\frac{6}{2}} = 2^3 = 8$$

To derive this relation, we could also use log

$$\log_{\sqrt{2}} 8 = \frac{\log 8}{\log \sqrt{2}} = \frac{3 \log 2}{\frac{1}{2} \log 2} = \frac{3}{\frac{1}{2}} = 6$$

Reminder:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a \cdot d}{b \cdot c} \Leftrightarrow \frac{a}{b} : \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

$$\frac{\frac{a}{b}}{c} = \frac{\frac{a}{1}}{\frac{b}{c}} = \frac{a \cdot c}{b}$$

$$\frac{\frac{a}{b}}{c} = \frac{\frac{a}{b}}{\frac{c}{1}} = \frac{a}{c \cdot b}$$