Lindström theorems for fragments of first-order logic

Balder ten Cate  Johan van Benthem  Jouko Väänänen

Abstract

Lindström theorems characterize logics in terms of model-theoretic conditions such as Compactness and the Löwenheim-Skolem property. Most existing Lindström theorems characterize extensions of first-order logic. On the other hand, many logics relevant to computer science are fragments or extensions of fragments of first-order logic, for example $k$-variable logics and various modal logics. Finding Lindström theorems for these languages can be a challenging problem, because most techniques used in the past rely on coding arguments that seem to require the full expressive power of first-order logic.

In this paper, we provide Lindström characterizations for a number of fragments of first-order logic. These include the $k$-variable fragments for $k > 2$, Tarski's relation algebra, graded modal logic, and the binary guarded fragment. We use two different proof techniques. One is a modification of the original Lindström proof, but with a new twist. The other involves the modal concepts of bisimulation, tree unraveling, and finite depth. Our results can also be used to derive semantic preservation theorems for these fragments of first-order logic.

Characterizing the 2-variable fragment or the full guarded fragment remain open problems.

1 Introduction

There are many ways to capture the expressive power of a logical language $\mathcal{L}$. For instance, one can characterize $\mathcal{L}$ as being a model theoretically well behaved fragment of a richer language $\mathcal{L}'$ (a preservation theorem), or as being maximally expressive while satisfying certain model theoretic properties (a Lindström theorem). The main contribution of this paper is a series of Lindström theorems for fragments of first-order logic. We also show connections between our Lindström theorems and preservation theorems.

The original Lindström theorem for first-order logic, in one of its formulations, says the following:

The (first-order) Lindström Theorem [8] An extension of first-order logic satisfies Compactness and the Löwenheim-Skolem property iff it is no more expressive than first-order logic.

There are several other versions of the theorem, characterizing first-order logic for instance in terms of Compactness and invariance for potential isomorphisms. Analogues of this result have been obtained for various extensions of first-order logic (e.g., [2, 5, 16]). On the other hand, few Lindström theorems are known for fragments of first-order logic. One notable example is Van Benthem’s recent Lindström theorem for modal logic:

The modal Lindström theorem [14] An extension of basic modal logic satisfies Compactness and bisimulation invariance iff it is no more expressive than basic modal logic.\(^1\)

Our general motivation for considering fragments comes from computer science logic. Many logics relevant to computer science are fragments (or extensions of fragments) of first-order logic, for example $k$-variable logics and various modal logics. Finding Lindström theorems for such languages can be a challenging problem, since most techniques used in the past to prove Lindström theorems rely on coding arguments that seem to require the full expressive power of first-order logic. For a recent account of Lindström theorem in a general setting, see [6].

We follow two global lines of attack. First, we take the original Lindström theorem for first-order logic and generalize the proof as much as possible. In this way, we obtain Lindström theorems for the finite variable fragments $FO^k$ with $k > 2$ and Tarski’s relation algebra. Next, we take the modal Lindström theorem as a starting point, and try to generalize it to richer languages. In this way, we obtain Lindström theorems for graded modal logic (on arbitrary Kripke structures and on trees) and the binary guarded fragment.

\(^1\)By extensions of basic modal logic we mean language extensions, not axiomatic extensions.
Many open question remain. For example, we have not been able to find Lindström theorems for the two-variable fragment or the full guarded fragment.

2 From first-order logic downwards

In this first part, we take the classic Lindström theorem as a starting point, and we generalize the argument to obtain characterizations for some fragments of first-order logic.

2.1 A strengthening of the Lindström theorem for first-order logic over binary vocabularies

The first-order Lindström theorem uniquely characterizes first-order logic in terms of Compactness and the Löwenheim-Skolem property within the class of all its extensions. As we will show in this section, this result can be improved: first-order logic can be characterized in terms of Compactness and the Löwenheim-Skolem property within the class of all extensions of the three-variable fragment $FO^3$, if we consider vocabularies consisting only of unary and binary relation symbols. As we will see in the next sections, this strengthening of the Lindström theorem allows us to obtain new results on Tarski’s relation algebra, as well as finite variable fragments.

To keep things simple, we will work with a fixed relational signature consisting of a set of unary relation symbols and a set of binary relation symbols, both countably infinite.

Important: We exclude relation symbols of arity $> 2$.

By an abstract logic we will mean a pair $\mathcal{L} = (\text{Fml}_\mathcal{L}, \models_\mathcal{L})$, where $\text{Fml}_\mathcal{L}$ is the set of sentences of $\mathcal{L}$ and $\models_\mathcal{L}$ is a binary relation between $\mathcal{L}$-sentences and models, indicating which sentences are true in which models. If no confusion arises, we will sometimes write $\mathcal{L}$ for $\text{Fml}_\mathcal{L}$ and $\models$ for $\models_\mathcal{L}$. We assume that $\mathcal{L}$-sentences are preserved under isomorphisms, and that $\mathcal{L}$ has the following properties:

- closure under Boolean connectives: for every $\phi \in \mathcal{L}$ there is a $\psi \in \mathcal{L}$ defining its negation (i.e., for all models $M, M \models \phi \iff M \not\models \phi$), and for every $\phi, \psi \in \mathcal{L}$ there is $\chi \in \mathcal{L}$ defining the conjunction of $\phi$ and $\psi$.

- closure under renamings: for every mapping $\rho$ sending relation symbols to relation symbols of the same arity, and for every sentence $\phi \in \mathcal{L}$, there is a sentence $\psi \in \mathcal{L}$ such that for all models $M, M \models \phi$ if $\rho(M) \models \psi$.

- closure under relativisation by unary predicates: for every sentence $\phi \in \mathcal{L}$ and unary relation symbol $P$, there is a sentence $\psi \in \mathcal{L}$ such that for all models $M, M \models \psi$ if $M^P \models \phi$, with $M^P$ the submodel of $M$ induced by $P$.

Examples of abstract logics include first-order logic ($FO$) and its $k$-variable fragment ($FO^k$), with $k \geq 1$.

Given two abstract logics, $\mathcal{L}$ and $\mathcal{L}'$, we say that $\mathcal{L}$ extends $\mathcal{L}'$ (or, $\mathcal{L}'$ is contained in $\mathcal{L}$, denoted by $\mathcal{L}' \subseteq \mathcal{L}$), if there is a map $f : \text{Fml}_{\mathcal{L}'} \rightarrow \text{Fml}_{\mathcal{L}}$ preserving truth in the sense that, for all models $M$ and sentences $\phi \in \mathcal{L}'$, $M \models \phi \iff M \models f(\phi)$.

An abstract logic $\mathcal{L}$ has Compactness if for every set of $\mathcal{L}$-formulas $\Sigma$, if every finite subset of $\Sigma$ is satisfiable then the entire set $\Sigma$ is satisfiable. An abstract logic $\mathcal{L}$ has the Löwenheim-Skolem property if every satisfiable set of $\mathcal{L}$-formulas has a countable model.

As a first step, we show that each compact extension of first-order logic, and in fact already of the two-variable fragment, has the “finite occurrence property”.

Lemma 2.1 (Finite occurrence property) Let $\mathcal{L}$ be an abstract logic extending $FO^2$ that has Compactness. Then for any $\phi \in \mathcal{L}$ there is a finite set of relation symbols $REL(\phi)$ such that the truth of $\phi$ in any model is independent of the denotation of relation symbols outside $REL(\phi)$.

The proof (a standard argument) is given in Appendix A. The heart of the proof of our improved Lindström theorem lies in the following lemma:

Lemma 2.2 Let $\mathcal{L}$ be an abstract logic with the Löwenheim-Skolem property, such that $\mathcal{L}$ extends $FO^2$ and is not contained in $FO$. Then “$\mathcal{L}$ can relatively projectively define finiteness”: there is a formula $\psi \in \mathcal{L}$ containing a unary predicate $N$, such that, for each $n \in \mathbb{N}$, there is a model of $\psi$ in which exactly $n$ elements satisfy $N$, while no model of $\psi$ has infinitely many elements satisfying $N$.

Proof: The basic idea is the same as in traditional proofs of the Lindström theorem (e.g., [5]). Our main contribution is to show that, in the case of binary vocabularies, the coding argument requires only three variables.

Take any $\phi \in \mathcal{L}$ not belonging to $FO$. Then for each $k \in \mathbb{N}$, there are models $\mathfrak{A}_k \models \phi$ and $\mathfrak{B}_k \not\models \phi$ that are potentially isomorphic up to back-and-forth depth $k$, while, at the same time, no potentially isomorphic models disagree on $\phi$. We can describe this situation inside $\mathcal{L}$. The construction is outlined in Figure 2.

The model depicted in Figure 2 describes two models, connected via a collection of partial isomorphisms, that disagree on $\phi$. The most important feature is that, if $N$ is an infinite set, then the collection of partial isomorphisms constitutes a potential isomorphism, whereas if $N$ is finite (say, of size $k$), the collection of partial isomorphisms constitutes a potential isomorphism up to back-and-forth depth $k$.

More precisely, $A$ and $B$ are unary predicates defining the domains of two (sub)models, $P$ is a unary predicate whose elements denote pairs from $A \times B$, and the elements of $F$ represent partial isomorphisms (i.e., sets of pairs constituting structure preserving bijections). The arrows represent a binary relation $R$. For instance, in the given example, $f$ represents the partial isomorphism...
Claim: Each of the following properties of this model can be expressed by a sentence of $L$:

1. Every $p \in P$ is associated to a pair from $A \times B$.
2. Every $f \in F$ is associated to a set of elements of $P$ that form a partial bijection between $A$ and $B$.
3. Each such partial bijection preserves structure on the submodels defined by $A$ and $B$, as far as the (finitely many) relations occurring in of $\phi$ are concerned.
4. Every $f \in F$ has a unique associated index taken from the set $N$.
5. $N$ is linearly ordered by $R$, such that there is a minimal element, and each non-maximal element has an immediate successor (in particular, if $N$ is infinite then it contains an infinite ascending chain).
6. If $f R g$ for $f, g \in F$, this means that $g$ extends $f$ (as a partial bijection), and that the index of $g$ is the successor of the index of $f$.
7. The back-and-forth properties hold for partial isomorphisms whose index is not the maximal element of $N$.
8. Some $f \in F$ has as index the minimal element of $N$.
9. The submodels defined by $A$ and $B$ disagree on $\phi$. (Recall that $L$ is closed under the Boolean connectives and relativisation by unary predicates).

Proof of claim: The first eight properties can already be expressed in $FO^3$ by a clever re-use of variables, and the ninth property can be expressed in $L$ by closure under the Boolean connectives and relativisation by unary predicates.

For instance, the fourth property is expressed as the conjunction of all $FO^3$-formulas of the following forms, for $S \in REL(\phi)$ a binary relation symbol, and $Q \in REL(\phi)$ a unary relation symbol.

$$\forall x \forall y (\exists z (Rxz \land Az \land \exists x (Ryx \land Ax \land Sxz)) \leftrightarrow \exists z (Rxz \land Bz \land \exists x (Ryx \land Bx \land Sxz)))$$

and

$$\forall x (\exists z (Rxz \land Az \land Qz) \leftrightarrow \exists z (Rxz \land Bz \land Qz))$$

Note that we crucially use the fact that the signature consists of unary and binary relations only. End of proof of claim.

Let $\chi$ be the conjunction of all these $L$-sentences. By construction, $\chi$ has models in which $N$ has arbitrarily large finite cardinality (this follows from the existence, for each $k \in \mathbb{N}$, of models disagreeing on $\phi$ that are potentially isomorphic up to back-and-forth depth $k$). However, there is no model of $\chi$ in which $N$ is an infinite set if there were, then, by the Löwenheim-Skolem property, there would be a countable such model, and in countable models, being potentially isomorphic means being isomorphic; thus, there would be isomorphic models disagreeing on $\phi$, which contradicts $L$'s invariance for isomorphisms). In other words, $\chi$ relatively projectively defines finiteness.

Theorem 2.3 An abstract logic extending $FO^3$ is contained in $FO$ iff it satisfies both Compactness and the Löwenheim-Skolem property.

Proof: If an abstract logic is contained in $FO$, then, clearly, it satisfies Compactness and the Löwenheim-Skolem property. If, on the other hand, an abstract logic $L$ extends $FO^3$ but is not contained in $FO$, then it must lack either the Löwenheim-Skolem property or Compactness. For, suppose $L$ satisfies the Löwenheim-Skolem property, and let $\psi(N)$ be any $L$-sentence projectively defining finiteness (cf. Lemma 2.2). Then every finite subset of $\{\exists x_1 \ldots x_k (N x_1 \land \cdots \land N x_k \land \bigwedge_{1 \leq i < j \leq k} (x_i \neq x_j)) \mid k \in \mathbb{N}\} \cup \{\psi(N)\}$ has a model while the entire set has no models. Thus, $L$ lacks compactness.

Note that this result relies on our restriction to unary and binary relation symbols. In the case with at most $k$-ary binary relations ($k > 2$) an analogous result holds for $FO^{k+1}$.

The following results can be proved in a similar fashion (relying again on the restriction to unary and binary relation symbols). We omit the details.

Figure 2. Model constructed in the proof of Lemma 2.2.

$\{ (a_1, b_1), (a_2, b_2), (a_3, b_3) \}$, and $g$ represents the partial isomorphism that extends $f$ with the pair $(a_4, b_1)$. The elements of the linearly ordered set $N$ are used as index of the partial isomorphisms.
Theorem 2.4 An abstract logic extending \( FO^3 \) is contained in \( FO \) iff it satisfies Compactness and invariance for potential isomorphisms.

Theorem 2.5 A “concrete” abstract logic extending \( FO^3 \) is contained in \( FO \) iff it satisfies the Löwenheim-Skolem property and is recursively enumerable for validity.

Here, by “concrete” we mean that formulas can be coded as finite strings over some alphabet, in such a way that negation, conjunction, and relativisation are computable operations, and there is a computable translation from \( FO^3 \)-formulas to formulas of the logic. The proof of Theorem 2.5 uses the fact that satisfiability of \( FO^3 \) formulas on finite models is undecidable [4].

2.2 First application: Tarski’s relation algebra

Tarski’s relation algebra \( RA \) [11] is an algebraic language in which the terms denote binary relations. It has atomic terms \( R, S, \ldots \) ranging over binary relations (over some domain), constants \( \delta \) and \( \top \) denoting the identity relation and the total relation, and operators \( \land, \lnot, \cdot, (\cdot)^\sim \) for taking the intersection, complement, composition and converse of relations. Thus, the syntax of \( RA \) is as follows:

\[
\alpha ::= R \mid \delta \mid \top \mid \alpha \land \beta \mid \lnot \alpha \mid \alpha \cdot \beta \mid \alpha^\sim
\]

with \( R \) an element from some countably infinite set of variables standing for binary relations. An interpretation for this language is a set \( X \) together with an assignment of binary relations over \( X \) to the atomic terms. In other words, it is a first-order structure for the vocabulary that contains the atomic terms as binary relation symbols. We write \( \alpha \equiv \beta \) if, in each interpretation, \( \alpha \) and \( \beta \) denote the same binary relation, and we write \( \alpha \subseteq \beta \) if, in each interpretation, \( \alpha \) denotes a subrelation of the relation denoted by \( \beta \).

In this section, we provide a Lindström-theorem for extensions of relation algebra. By an extended relation algebra we will mean any language obtained by extending the syntax of relation algebra with zero or more additional logical operations, where a logical operation is any operation that takes as input a fixed finite number of binary relations \( R_1, \ldots, R_n \) (over some set \( X \)), and that produces a new binary relation \( S \) over the same set. We also require logical operations to respect isomorphisms, and to be domain independent in the following sense (familiar from database theory): the output relation \( S \) does not depend on elements of the domain \( X \) that do not participate in any pair belonging to any input relation \( R_i \). This excludes for instance complementation as a logical operation, but, in terms of expressivity, it is not an essential restriction: one can always relativise such operations, by introducing an additional argument. For instance, absolute complementation as in \( \lnot R \) may be replaced by relative complementation as in \( \top - R \).

One example of an extended relation algebra, \( RA_{FO} \), is the extension of relation algebra with all first-order definable logical operations (see e.g. [15]). Another example is \( RA_T \), the extension of relation algebra with the transitive closure operation [9].

The compactness and Löwenheim-Skolem properties can be defined for extended relation algebras as usual. For instance, we say that an extended relation algebra \( L \) has the Löwenheim-Skolem property if every set of \( L \)-expressions \( \Phi \), if there is an interpretation under which \( \bigcap \Phi \) is a non-empty relation, then there is such an interpretation over a countable domain. As is not hard to see, \( RA \) and \( RA_{FO} \) satisfy both Compactness and the Löwenheim-Skolem property, whereas \( RA_T \) satisfies the Löwenheim-Skolem property but lacks Compactness.

The following Lindström-style theorem shows that all extended relation algebras containing non-first-order definable operations lack either Compactness or the Löwenheim-Skolem property.

Theorem 2.6 Let \( L \) be any extended relation algebra with the Compactness and Löwenheim-Skolem properties. Then every logical loperation of \( L \) is first-order definable.

Proof: Lemma 2.2 can be adapted to the relation algebra setting, allowing us to show that every extended relation algebra containing a non-elementary logical operation and having the Löwenheim-Skolem property can projectively define finiteness, and hence lacks Compactness. We will not spell out the details, but merely mention the following key points of the proof:

- For every first-order sentence \( \phi \) containing only three variables, in a signature consisting only of binary relations, there is a relation algebra expression \( R \) such that the two are equivalent in the following sense: for every model \( M \), \( M \models \phi \) iff \( R \equiv \top \) holds in \( M \) [11].
- Unary relations can be mimicked by binary relations, for instance by systematically intersecting them with the identity relation.
- In this way, every extended relation algebra gives rise to an abstract logic extending \( FO^3 \). Closure under relativisation of the logic is guaranteed by the domain independence of the logical operations of \( L \).

In other words, the extension of relation algebra with all elementary operations is the greatest extension that satisfies Löwenheim-Skolem and Compactness. The same holds if we replace the Löwenheim-Skolem property by invariance for potential isomorphisms, or if we replace Compactness by recursive enumerability.

Theorem 2.6 nicely complements a known result: every extended relation algebra with Craig interpolation can define all first-order definable operations [12]. Together, these results show that \( RA_{FO} \) is the unique (up to expressive equivalence) extension of \( RA \) satisfying Compactness, Löwenheim-Skolem, and Craig Interpolation.
2.3 Second application: finite variable fragments over binary vocabularies

In this section, we provide Lindström theorems for the finite variable fragments FO\(^k\) with \(k \geq 3\), over vocabularies consisting of unary and binary relation symbols only. It is well known that the finite variable fragments can be characterized as fragments of first-order logic using potential isomorphisms with a restricted number of pebbles:

**Definition 2.7 (k-pebble potential isomorphisms)** A k-pebble potential isomorphism between M and N is a non-empty family \(F\) of finite partial isomorphisms \(f\) between \(M\) and \(N\) with \(|\text{dom}(f)| \leq k\) that satisfies the following back-and-forth properties:

- **Forth:** for all \(f \in F\), \((w_1, v_1), \ldots, (w_n, v_n) \in f\) with \(n < k\), and \(w, v \in M\), there is an \(v' \in N\) such that \(\{(w_1, v_1), \ldots, (w_n, v_n), (w, v')\} \in F\)
- **Back:** for all \(f \in F\), \((w_1, v_1), \ldots, (w_n, v_n) \in f\) with \(n < k\), and \(v \in N\), there is an \(w \in M\) such that \(\{(w_1, v_1), \ldots, (w_n, v_n), (w, v)\} \in F\)

**Fact 2.8 (Folklore)** FO\(^k\) is (up to logical equivalence) the fragment of FO invariant for k-pebble potential isomorphisms (\(k \geq 1\)).

Using Lemma 2.2, we can turn this into the following Lindström characterization (remember that we only consider unary and binary relation symbols):

**Theorem 2.9 (Lindström theorem for FO\(^k\))** Let \(k \geq 3\). An abstract logic extending FO\(^3\) satisfies Compactness and invariance for k-pebble potential isomorphisms iff it is no more expressive than FO\(^k\).

**Proof:** Consider any abstract logic \(\mathcal{L}\) extending FO\(^3\) that has Compactness and is invariant for k-pebble potential isomorphisms. Then in particular it is invariant for potential isomorphisms, and therefore by Theorem 2.4 it must be contained in FO. But then, by Fact 2.8, it must be contained in FO\(^k\).

Theorem 2.9 can be seen as a strengthening of Fact 2.8 (for \(k \geq 3\), and on binary vocabularies). Indeed, it implies the following “generalized preservation theorem” (again, with only unary and binary relation symbols):

**Corollary 2.10** Let \(k \geq 3\). Let \(\mathcal{L}\) be any abstract logic extending FO\(^k\) with Compactness. Then FO\(^k\) is the fragment of \(\mathcal{L}\) invariant for k-pebble potential isomorphisms (up to logical equivalence). In particular, FO\(^k\) is the fragment of FO invariant for k-pebble potential isomorphisms.

**Proof:** Let \(\mathcal{L}\) be any compact abstract logic extending FO\(^k\), and let \(\mathcal{L}'\) be the fragment of \(\mathcal{L}\) invariant for k-pebble potential isomorphisms. Then \(\mathcal{L}'\) satisfies all requirement of abstract logics. For instance, it is closed under relativisation: if \(\phi \in \mathcal{L}\) is invariant for k-pebble potential isomorphisms, then so is its relativisation by a unary predicate. Likewise for the Boolean connectives. Thus, \(\mathcal{L}'\) is an abstract logic extending FO\(^k\) that is compact and invariant for k-pebble potential isomorphisms. Hence, by Theorem 2.9, it is contained in FO\(^3\). \(\square\)

3 From modal logic upwards

The approach of generalizing the classic Lindström theorem only got us so far. It enabled us to characterize FO\(^k\) for \(k \geq 3\) but is unlikely to reach much further down. We will now take a different approach, by considering the modal Lindström theorem, and trying to generalize it to richer languages. In particular, we obtain two Lindström theorems for the graded modal logic.

3.1 The modal Lindström theorem revisited

We recall the proof of the modal Lindström theorem of [14] (which improves on an earlier result in [10]). First, we need to define “abstract modal logics”. We will again assume a fixed vocabulary, this time consisting of a single binary relation symbol \(R\) and a countably infinite set of unary relation symbols (also called proposition letters). Structures for this vocabulary are usually called Kripke models (the restriction to a single binary relation symbol is not essential but is convenient for presentational reasons). We associate to each formula a class of pairs \((M, w)\), where \(M\) is a Kripke model and \(w\) is an element of the domain of \(M\). This is because modal formulas are always evaluated at a point in a model. We will call such pairs \((M, w)\) pointed Kripke models. Thus, an abstract modal logic is a pair \(\mathcal{L} = (\text{Fml}_\mathcal{L}, \models)\), where \(\text{Fml}_\mathcal{L}\) is the set of formulas of \(\mathcal{L}\) and \(\models\) is a binary relation between \(\mathcal{L}\)-formulas and pointed Kripke models. As before, when no confusion arises we will write \(\mathcal{L}\) for \(\text{Fml}_\mathcal{L}\). Also, as before, we assume that \(\mathcal{L}\)-formulas are invariant for isomorphisms, and that \(\mathcal{L}\) is closed under the Boolean operations, renaming, and relativisation by unary predicates.

Examples of abstract modal logics include basic modal logic, its extension with counting modalities called graded modal logic (GML), first-order logic (by which we mean the collection of all first-order formulas in one free variable, over the appropriate signature), and the modal \(\mu\)-calculus. For the syntax and semantics of basic modal logic, the reader may consult any modal logic textbook (e.g., [3]).

The modal Lindström theorem characterizes basic modal logic in terms of Compactness and invariance for bisimulations. A bisimulation between Kripke models \(M\) and \(N\) is a binary relation \(Z\) between the domains of \(M\) and \(N\) satisfying the following three conditions:

- Atomic harmony: if \(wZv\) then \(w\) and \(v\) agree on all proposition letters (i.e., unary predicates).
- Zig: if \( wZv \) and \( wR^Mw' \), there is a \( v' \) such that \( vR^Nv' \) and \( wZv' \).

- Zag: if \( wZv \) and \( vR^Nv' \), there is a \( w' \) such that \( wR^Mw' \) and \( wZv' \).

Two pointed Kripke models \( (M, w) \) and \( (N, v) \) are said to be bisimilar if there is a bisimulation \( Z \) between \( M \) and \( N \) such that \( wZv \). A formula is bisimulation invariant if it does not distinguish bisimilar pointed Kripke models, and an abstract modal logic is bisimulation invariant if all its formulas are.

Given a pointed Kripke model \( (M, w) \), we denote by \( M_w \) the submodel of \( M \) containing all points that are reachable in finitely many steps from \( w \) along the binary relation. Likewise, for \( k \in \mathbb{N} \), \( M^k_w \) is the submodel of \( M \) containing all points reachable from \( w \) in at most \( k \) steps along the binary relation. We say that a formula \( \phi \) is invariant for generated submodels if, for all models \( M \) with worlds \( w, (M, w) \models \phi \) iff \( (M_w, w) \models \phi \). We say that \( \phi \) has the finite depth property if there is a \( k \in \mathbb{N} \) such that \( (M, w) \models \phi \) iff \( (M^k_w, w) \models \phi \), for all models \( M \) with worlds \( w \). Clearly, the latter implies the former. Also, bisimulation invariance implies invariance for generated submodels, because the (graph of the) natural inclusion map is a bisimulation. An abstract modal logic \( \mathcal{L} \) is said to be invariant for generated submodels (or, to have the finite depth property), if every \( \phi \in \mathcal{L} \) is invariant for generated submodels (respectively, has the finite depth property).

We are now ready to proceed with the proof of Theorem 3.3. We first prove a finite occurrence property for compact extensions of basic modal logic that satisfy invariance for bisimulation (in fact, we only need to assume invariance for generated submodels).

**Lemma 3.1 (Finite occurrence property)** Let \( \mathcal{L} \) be any abstract modal logic extending basic modal logic that is Compact and invariant for generated submodels. Then for each \( \phi \in \mathcal{L} \) there is a finite set of proposition letters \( PROP(\phi) \) such that the truth of \( \phi \) in any pointed Kripke model is independent of the denotation of proposition letters outside \( PROP(\phi) \).

The proof is given in Appendix A.

The heart of the proof of Theorem 3.3 is in the following:

**Lemma 3.2** Let \( \mathcal{L} \) be any abstract modal logic extending basic modal logic that is Compact and invariant for generated submodels. Then every \( \mathcal{L} \) has the finite depth property.

**Proof:** Take any \( \phi \in \mathcal{L} \), and let \( p \) be a proposition letter not occurring in \( \phi \). By the generated submodel-invariance of \( \mathcal{L} \), \( \{p, \Box p, \Diamond p, \ldots\} \models \phi \leftrightarrow \Box^p \phi \). By the compactness of \( \mathcal{L} \), there is an \( n \in \mathbb{N} \) such that \( \{p, \Box p, \ldots, \Box^n p\} \models \phi \leftrightarrow \Box^p \phi \). But this expresses exactly that \( \phi \) has the finite depth property, for depth \( n \).

Now follows easily:

**Theorem 3.3 ([14])** An abstract modal logic extending basic modal logic satisfies Compactness and bisimulation invariance iff it is no more expressive than basic modal logic.

**Proof:** Let \( \mathcal{L} \) be any abstract modal logic extending basic modal logic and satisfying Compactness and bisimulation invariance. Since bisimulation invariance implies invariance for generated submodels, \( \mathcal{L} \) is also invariant for generated submodels. Hence, by Lemma 3.1 and Lemma 3.2, \( \mathcal{L} \) has the finite occurrence property and the finite depth property. Next, we use the following well known fact [3]:

Assuming a finite vocabulary, every bisimulation-invariant class of pointed models with the finite depth property is definable by a formula of basic modal logic.

We conclude that \( \mathcal{L} \) is contained in basic modal logic. \( \square \)

Theorem 3.3 can be seen as a strengthening of the more familiar characterization of basic modal logic as the bisimulation invariant fragment of first-order logic. Indeed, it implies the following “generalized preservation theorem”:

**Corollary 3.4** Let \( \mathcal{L} \) be any abstract modal logic extending basic modal logic that has Compactness. Then basic modal logic is the bisimulation invariant fragment of \( \mathcal{L} \) (up to logical equivalence). In particular, basic modal logic is the bisimulation invariant fragment of first-order logic.

**Proof:** Let \( \mathcal{L}' \) be the bisimulation invariant fragment of \( \mathcal{L} \). Then \( \mathcal{L}' \) satisfies all criteria for being an abstract modal logic. For instance it is closed under relativisation: whenever \( \phi \in \mathcal{L} \) is invariant for bisimulations then the relativisation of \( \phi \) by a unary predicate is also invariant for bisimulations. Likewise for the Boolean connectives.

Hence, \( \mathcal{L}' \) is an abstract modal logic extending basic modal logic and it is bisimulation invariant and Compact. Hence, it is no more expressive than basic modal logic. \( \square \)

Corollary 3.4 is really stronger than the traditional bisimulation preservation theorem, because there are compact extensions of basic modal logic that are not contained in first-order logic. Indeed:

**Theorem 3.5** There is an abstract modal logic extending basic modal logic that is not contained in first-order logic, but still satisfies Compactness, the Löwenheim-Skolem property, invariance for potential isomorphisms, invariance for generated submodels, and finite axiomatizability.

The proof is given in Appendix B. Theorem 3.5 is also interesting for another reason: it shows that Theorem 2.3 no longer holds when \( FO^3 \) is replaced by basic modal logic.
3.2 Graded modal logic

Graded modal logic (GML) extends basic modal logic with counting modalities: for each formula $\phi$ and natural number $k$, $\Diamond_k \phi$ is admitted as a formula, and it says that at least $k$ successors of the current node satisfy $\phi$.

GML-formulas are in general not invariant for bisimulations. Still, an important (weaker) invariance property does hold: GML formulas are invariant for tree unrolling. A tree model is a Kripke model whose underlying frame is a tree (possibly infinite, but well-founded, and with a unique root). We will denote tree models by $T, T', \ldots$, and we will use $\text{root}(T)$ to denote the root of the tree model $T$. Every pointed Kripke model $(M, w)$ can be unraveled into a tree model, by the following standard construction:

**Definition 3.6 (Tree unraveling)** Given a Kripke model $M = (W, R, V)$ and $w \in W$, the tree unraveling $\text{unr}(M, w)$ is defined as $(W', R', V')$, where

- $W'$ consists of all finite paths $\langle w_1, w_2, \ldots, w_n \rangle$ satisfying $w_1 = w$ and $w_i R w_{i+1}$.
- $R'$ contains all pairs of sequences of the form $\langle \langle w_1, \ldots, w_n \rangle, \langle w_1, \ldots, w_n, w_{n+1} \rangle \rangle \in W' \times W'$.
- $\langle w_1, \ldots, w_n \rangle \in V'(p)$ iff $w_n \in V(p)$.

It is easily seen that, for any pointed Kripke model $(M, w)$, $\text{unr}(M, w)$ is indeed a tree model, and that $(w)$ is its root. GML-formulas are invariant for this operation:

**Fact 3.7 (GML is invariant for tree unrollings)** For all pointed Kripke models $(M, w)$ and GML-formulas $\phi$, $M, w \models \phi$ iff $\text{unr}(M, w), \langle w \rangle \models \phi$.

We will prove two Lindström theorems for GML. The first characterizes GML on arbitrary structures in terms of Compactness, the Löwenheim-Skolem property and invariance for tree unrollings. It can be seen as a natural generalization of Theorem 3.3. The second theorem, which will be proved in the following section, considers GML as a language for describing (nodes in) tree models, and characterizes GML as being maximal with respect to Compactness and the Löwenheim-Skolem property on such structures.

Recall Theorem 3.3, which characterizes modal logic in terms of Compactness and bisimulation invariance. One might wonder if, likewise, GML can be characterized in terms of Compactness and invariance for tree unrolling. The answer is negative. In particular, the extension of GML with the modal operator $\Diamond_{\omega}$ (“there are uncountably many successors . . .”) still satisfies these properties. Instead, we will prove the following:

**Theorem 3.8** An abstract modal logic extending GML satisfies invariance for tree unrollings, Compactness, and the Löwenheim-Skolem property iff it is no more expressive than GML.

As in the case of modal logic, we obtain the following “generalized preservation theorem” as a corollary (the proof is analogous to that of Corollary 3.4):

**Corollary 3.9** Let $\mathcal{L}$ be any abstract modal logic extending GML and satisfying Compactness and the Löwenheim-Skolem property. Then GML is the tree unraveling invariant fragment of $\mathcal{L}$ (up to logical equivalence). In particular, GML is the tree unraveling invariant fragment of FO.

The rest of this section is dedicated to the proof of Theorem 3.8. Two facts about GML will be used in the proof:

**Fact 3.10 (GML has the finite tree model property)** Every satisfiable GML formula is satisfied at the root of a finite tree model.

**Fact 3.11 (GML can describe finite tree models up to isomorphism)** Assuming a finite vocabulary, for every finite tree model $T$ there is a GML-formula $\psi_T$ such that for every tree model $T'$, $(T', \text{root}(T')) \models \psi_T$ iff $T' \cong T$.

Now for the main argument. Fix an abstract modal logic $\mathcal{L}$ extending GML and satisfying Compactness and Löwenheim-Skolem, and invariance for tree unrollings. By Lemma 3.1 and 3.2, $\mathcal{L}$ has the finite occurrence property and the finite depth property (note that invariance for tree unraveling implies generated submodel invariance). The following Lemma shows a kind of “finite width property”.

**Lemma 3.12 ($\mathcal{L}$-formulas can only count successors up to a finite number)** For each formula $\phi \in \mathcal{L}$ and finite tree model $T$, there is a natural number $k$ such that “$\phi$ can only count $T$-successors up to $k$”: whenever a tree model contains a node $v$ that has $k$ successor subtrees isomorphic to $T$, then adding more copies of $T$ will not affect the truth value of $\phi$ at the root.

**Proof:** Since $T$ is a finite tree model, it can be described up to isomorphism by a single GML-formula $\psi_T$. Let $\Sigma$ be the following set of formulas, where $p$ is a proposition letter not occurring in $\phi$, and $\square^n$ stands for a sequence of $n$ boxes:

\[
\{ p, \square^n(\neg p \rightarrow \square^n p) \mid n \in \mathbb{N} \}
\]

“$p$ defines an initial subtree”

\[
\{ \square^n(p \rightarrow \square(\neg p \rightarrow \psi_T)) \mid n \in \mathbb{N} \}
\]

“the root of every $\neg p$-subtree satisfies $\psi_T$”

\[
\{ \square^n(p \land \Diamond_{\omega} \neg p \rightarrow \Diamond_m (p \land \psi_T)) \mid n, m \in \mathbb{N} \}
\]

“every $p$-node with a $\neg p$-successor has infinitely many $p$-successors satisfying $\psi_T$”

Whenever a countable tree model satisfies $\Sigma$ at the root, the submodel defined by $p$ is isomorphic to the whole model — isomorphic in the language without $p$, to be precise (see Figure 3). Since $\Sigma$ satisfies the Löwenheim-Skolem property, invariance for tree unrolling and invariance for isomorphisms, we can conclude that $\Sigma \models \phi \leftrightarrow \phi^p$. But then,
by compactness, there is a \( k \in \mathbb{N} \) such that \( \Sigma_k \models \phi \leftrightarrow \phi^p \), where \( \Sigma_k \) is the following subset of \( \Sigma \):
\[
\{ p, \Box^n (\neg p \rightarrow \Box \neg p) \mid n \in \mathbb{N} \} \cup \\
\{ \Box^n (p \rightarrow \Box (p \lor \psi_T)) \mid n \in \mathbb{N} \} \cup \\
\{ \Box^n (p \land \Box \neg p \rightarrow \Diamond^m (p \land \psi_T)) \mid n, m \in \mathbb{N} \text{ such that } m \leq k \}
\]
This shows that the Lemma holds. \( \square \)

**Proof of Theorem 3.8:** Let \( \mathcal{L} \) be any abstract modal logic extending GML, satisfying Compactness, the Löwenheim-Skolem property, and tree unraveling invariance. Observe that \( \mathcal{L} \) still satisfies Compactness and the Löwenheim-Skolem property if we restrict attention to trees (note that the tree unraveling of a countable model is countable).

Consider any formula \( \phi \in \mathcal{L} \). We will construct a set of equivalence relations \( \sim_i^\phi \) for tree models (with \( i \geq 0 \)), satisfying the following two properties:

1. \( T \sim_i^\phi T' \) implies that the truth value of \( \phi \) at the root of a tree model is not affected if subtrees isomorphic to \( T \) at depth \( i \) are replaced by copies of \( T' \) (or vice versa).
2. Each \( \sim_i^\phi \) has only finitely many equivalence classes, and each is definable by a GML-formula.

This then implies that \( \phi \) is equivalent to a GML formula (take the disjunction of the GML-formulas defining the \( \sim_0^\phi \)-equivalence classes that satisfy \( \phi \)).

We will prove the claim by induction. It holds trivially for \( i \geq \text{depth}(\phi) \). Next, assume that the claim holds for \( i + 1 \). We will show that it also holds for \( i \). Let \( K_1, \ldots, K_n \) be the (finitely many) \( \sim_{i+1}^\phi \)-equivalence classes, and for each \( \ell \leq n \), pick a finite representative \( T_\ell \in K_\ell \) (using Proposition 3.10). It follows from Lemma 3.12, there is a \( k \in \mathbb{N} \) such that, for all \( \ell \leq n \), \( \phi \) can only count \( k \)-successors up to \( k' \), and hence, by \( \sim_{i+1}^\phi \)-equivalence, \( \phi \) can only count \( K_\ell \)-successors up to \( k' \), at depth \( i' \). But then, it follows that there are at most \( k^{n \cdot 2^{\text{prop}(\phi)}} \) many \( \sim_i^\phi \)-equivalence classes. Moreover, they are definable by GML-formulas (in fact, by Boolean combinations of proposition letters and formulas of the form \( \Diamond^m \psi \) with \( m \leq k \) and \( \phi \) the GML-formula defining some \( \sim_{i+1}^\phi \)-equivalence class).

Thus, \( \mathcal{L} \) is not more expressive than GML on tree models. It follows that \( \mathcal{L} \) is not more expressive than GML on arbitrary Kripke models: consider any \( \mathcal{L} \)-formula \( \phi \), and let \( \psi \) be any GML-formula equivalent to \( \phi \) on tree models. If \( \phi \leftrightarrow \psi \) were falsifiable, then, by unraveling, it could be falsified on a tree, which, by assumption, is not the case. Thus, \( \phi \) and \( \psi \) are equivalent on all Kripke models. \( \square \)

### 3.3 Graded modal logic on trees

In this section, rather than assuming tree unraveling invariance, we consider only tree models from the start. That is, we view GML as a language for describing nodes of tree models. From this perspective, GML has three distinctive limiting features: (i) when evaluated in a node, formulas can only see the subtree starting from that node; (ii) when evaluated at a node, each formula can only look finitely deeply into the subtree starting at that node; (iii) each formula allows us to uniformly substitute formulas for proposition letters under relativisation on trees, if for every formula \( \phi \in \mathcal{L} \) and proposition letter \( p \), there is a formula \( \psi \in \mathcal{L} \) such that for all pointed tree models \( (T, n, p) \), \( (T, n) \models \psi \iff (T, n) \models p \) and \( (\text{Subtree}(T, n, p), n) \models \phi \). In the case of GML, we can simply pick \( \psi \) to be the syntactic relativisation of \( \phi \) by \( p \), i.e., the formula obtained from \( \phi \land p \) by replacing all subformulas of the form \( \Diamond^m \psi \) by \( \Diamond^m (p \land \psi) \).

Secondly, for the proof we need to make an extra assumption, namely that the extensions \( \mathcal{L} \) we consider are closed under substitution. Intuitively, this means that \( \mathcal{L} \) allows us to uniformly substitute formulas for proposition letters. More precisely, \( \mathcal{L} \) is closed under substitution if for all formulas \( \phi, \psi \in \mathcal{L} \) and proposition letters \( p \), there is a formula \( \chi \) such that for all pointed (tree) models \( (M, w), (M, w) \models \chi \iff (M[p \leftarrow \psi][M, v \models \chi], w) \models \phi \).
Proof:
Since \( n \) is satisfiable: it is true at \((T, n)\) when we make \( p \) true at all nodes, and \( q \) only at \( n \) and its descendants. We claim that truth \( \Sigma \) at a node in a tree implies that \( n \) has a parent and it satisfies either \( p \) or \( q \). For, if not then the submodels \( \text{Subtree}(T, n, p) \) and \( \text{Subtree}(T, n, q) \) would coincide, and hence \( \phi^p \) and \( \phi^q \) would have to have the same truth value at \( n \).

Next, we will use Compactness to obtain a finite subset of \( \Sigma \) that implies that the current node has a parent satisfying \( p \lor q \). First, we ‘redescribe’ the situation encoded by \( \Sigma \) from the perspective of the parent node. Let \( \Sigma' \) be the following set of \( \mathcal{L} \)-formulas, where \( r \) is another fresh proposition letter:

\[
\Sigma' = \{ \diamond (\phi^p \land \phi^q \land r), \Box (r \lor \Box^n (p \land q)) \mid n \geq 0 \}
\]

By the previous observations, \( \Sigma' \models p \lor q \). Hence, by Compactness, there is a \( k \in \mathbb{N} \) such that \( \diamond (\phi^p \land \phi^q \land r) \land \bigwedge_{n \leq k} \Box (r \lor \Box^n (p \land q)) \models p \lor q \). Going back to the perspective of the node \( n \), if we define \( \psi \) to be the formula \( \{ \phi^p \land \phi^q \land \bigwedge_{n \leq k} \Box^n (p \land q) \} \) then \( \psi \) is satisfiable and implies the existence of a parent node satisfying \( p \lor q \).

Finally, we take two more fresh proposition letters, \( s \) and \( t \), and we use the fact that \( \mathcal{L} \) is closed under substitution: we define \( \chi \) to be \( s \land \psi[p/(p \land (s \land t)), q/(q \land (s \land t))] \).

On the one hand, truth of \( \chi \) at a node implies that it has a parent satisfying either \( (p \land (\diamond s \land t)) \) or \( (q \land (\diamond s \land t)) \), and hence \( \psi \). On the other hand, there is a pointed tree satisfying \( \chi \) in which \( t \) is only true at the parent node: it suffices to consider \((T, n)\) and extend the valuation by making \( s \) true only at \( n \), and \( t \) at its parent.

Lemma 3.16 \( \mathcal{L} \) is invariant for generated submodels

Proof: Suppose not. Let \( \chi(p) \in \mathcal{L} \) be as described by Lemma 3.15. By a “fresh renaming” of \( \chi \) we will mean a copy in which all proposition letters have been renamed to fresh ones, and which has been relativised by an additional fresh proposition letter. For the reasons explained in the proof of Lemma 3.15, we may assume that \( \chi(p) \) has infinitely many fresh renamings \( \langle \chi_i(p_i) \rangle_{i \in \mathbb{N}} \).

Finally, we define \( \Sigma \) to be the set of \( \mathcal{L} \)-formulas \( \{ \chi_1(p_1), \chi_1(\chi_2(p_2)), \chi_1(\chi_2(\chi_3(p_3))), \ldots \} \). Every finite subset of \( \Sigma \) is satisfiable. Indeed, a satisfying model may be constructed by “overlaying” different copies of the model \((T, n)\) from Lemma 3.15(2). On the other hand, if a node would satisfy all formulas in \( \Sigma \) at once, its ancestors would form an infinite ascending chain, which contradicts the well-foundedness property of trees.

3.4 Binary guarded fragment

The guarded fragment \( \mathcal{GF} \) forms a second extension of modal logic, incomparable to graded modal logic. It allows for arbitrary quantifications of the form \( \exists \vec{x} \forall \vec{y} \, (G(\vec{x}, \vec{y}) \land \phi(\vec{x}, \vec{y})) \), where \( \vec{x} \) and \( \vec{y} \) are tuples of variables, and the
guard $G$ is an atomic formula containing the variables in $\vec{x}$ and $\vec{y}$. The guarded fragment was proved to be decidable and to have many ‘modal’ meta-properties, thanks to its invariance for guarded bisimulations [1, 13], see Appendix C.

Because of its modal character, GF seems an obvious case for a Lindström-style analysis like the one we have given for modal logic and graded modal logic. However, there are some technical difficulties, and we have not been able to prove a Lindström theorem for this language yet. In this section, we focus on a special case, in fact the same special case as in the first half of the paper, namely for vocabularies with only unary and binary predicates. The Binary Guarded Fragment (GFbin) has the following syntax:

$$\phi ::= R\vec{x} \mid x = y \mid \neg \phi \mid \phi \lor \psi \mid \exists \vec{y}(G(\vec{x}, \vec{y}) \land \psi(\vec{x}, \vec{y}))$$

where $\vec{x}$ and $\vec{y}$ are non-empty and non-overlapping tuples of variables, and the guard $G$ is an atomic formula containing the variables in $\vec{x}$ and $\vec{y}$ (in any order and multiplicity) with the same restrictions as before, and where all atomic predicates are unary or binary. GFbin is easily seen to be an abstract logic contained in $FO^2$. When interpreted over Kripke structures (and considering only formulas with one free variable), it constitutes an abstract modal logic extending basic modal logic.

**Theorem 3.17** An abstract logic extending the GFbin satisfies Compactness and Invariance for Guarded Bisimulations iff it is no more expressive than GFbin.

The proof follows the same general lines as in the case of modal logic and graded modal logic: using Compactness, we prove a finite occurrence property and a finite depth property (where depth is now measured as distance in the Gaifman graph). We then use a tree-unraveling argument to show that GFbin can express all properties invariant for guarded bisimulations and having the finite depth property. It is exactly in this last step that the restriction to unary and binary relation symbols is crucial (roughly, it allows us to relate distance in the unravelled tree to Gaifman distance in the original structure). More details are in Appendix C.

### 4 Open questions

To conclude, we identify three lines of future research.

**Strengthening the characterizations.** A natural question is whether Theorem 2.3 can be improved. For instance, are there any extensions of $FO^2$ not contained in $FO$ that satisfy Compactness and the Löwenheim-Skolem property? Note that there are extensions of modal logic not contained in first-order logic with these properties (cf. Theorem 3.5).

**Characterizing more languages.** Another line of open problems is to find Lindström characterizations for other fragments, such as $FO^2$ and the guarded fragment. Also, we can consider non-elementary extensions of modal logic such as the modal $\mu$-calculus. For instance, can one characterize the modal $\mu$-calculus in terms of bisimulation invariance and the finite model property?

**Characterizing logics on specific classes of structures.** No Lindström characterizations are known for first-order logic on finite structures, or on trees. Compactness fails for first-order logic on such structures, and, on finite structures, the Löwenheim-Skolem property becomes meaningless. In this paper, we proved one positive result: we showed that GML behaves on trees as first-order logic does on arbitrary structures: it is maximal with respect to Compactness and the Löwenheim-Skolem property. In general, however, this area remains underexplored [7].

### References


A Some missing proofs

Proof of Lemma 2.1: Cf. [5]: since our vocabulary contains infinitely many unary and binary relation symbols, there are renamings \( \rho_1, \rho_2 \) whose range is disjoint. Take any \( \phi \in \mathcal{L} \), and let \( \phi_1, \phi_2 \) be its renamings according to \( \rho_1 \) and \( \rho_2 \). Let \( \Sigma = \{ \forall x_1 \ldots x_k (\rho_1(R)(x_1 \ldots x_k) \leftrightarrow \rho_2(R)(x_1 \ldots x_k)) \mid R \text{ a k-ary relation symbol} \} \). Then \( \Sigma \models \phi_1 \leftrightarrow \phi_2 \), and hence, by Compactness, a finite subset \( \Sigma' \subseteq \Sigma \) implies \( \phi_1 \leftrightarrow \phi_2 \). We can pick for REL(\( \phi \)) the (finite) set of relation symbols occurring in \( \Sigma' \).

Proof of Lemma 3.1: Like that of Lemma 2.1. Since the set of proposition letters (unary predicates) is infinite, we can find renamings \( \rho_1, \rho_2 \) for them whose range is disjoint. Now, take any \( \phi \in \mathcal{L} \), and let \( \phi_1, \phi_2 \) be its renamings according to \( \rho_1 \) and \( \rho_2 \). Let \( \Sigma = \{ \forall^n(\rho_1(p) \leftrightarrow \rho_2(p)) \mid n \in \mathbb{N} \text{ and } p \text{ a proposition letter} \} \), where \( \forall^n \) stands for a sequence of \( n \) boxes. It follows from bisimulation invariance that \( \Sigma \models \phi_1 \leftrightarrow \phi_2 \), hence, by Compactness, a finite subset \( \Sigma' \subseteq \Sigma \) implies \( \phi_1 \leftrightarrow \phi_2 \). We can pick for PROP(\( \phi \)) the set of proposition letters occurring in \( \Sigma' \).

B A well-behaved non-elementary extension of modal logic

Definition B.1 \( ML^* \) is obtained by adding an extra operator \( \bullet \) to basic modal logic. Thus, the formulas of \( ML^* \) are given by:

\[
\phi ::= p \mid \neg \phi \mid \phi \land \psi \mid \Diamond \phi \mid \bullet \phi
\]

The semantics of the newly added operator is as follows: \( M, w \models \bullet \phi \) if \( w \) has infinitely many reflexive successors satisfying \( \phi \) (a reflexive world is one related to itself).

Theorem B.2 \( ML^* \) is an abstract modal language with Compactness, the Löwenheim-Skolem property, invariance for potential isomorphisms, and a finite axiomatization, but it is not contained in FO.

Proof: \( ML^* \) is indeed an abstract modal logic (in particular, is closed under relativisation), and it is not contained in FO. The invariance for potential isomorphisms and the Löwenheim-Skolem property follow from the fact that \( ML^* \) is contained in \( L_{\psi,w} \). It only remains to be shown that \( ML^* \) is compact and that it has a finite axiomatization.

We will provide a finite axiomatization that is sound and strongly complete (i.e., every consistent set of formulas is satisfied in some model). It extends the modal system K with four additional axioms (cf. Table 1).

Soundness is left as an exercise. For completeness, we use a variant of the canonical model construction (a standard construction in modal logic [3]): we will construct a model out of maximal consistent sets of formulas (“MCSs”), such that a formula is true at a world iff it belongs to the corresponding MCS. It then follows that every consistent set of formulas is satisfiable.

As domain \( W \), we choose the set of all pairs \( (w, n) \), where \( w \) a maximal consistent set of \( ML^* \) formulas and \( n \) a natural number. This will ensure that each satisfiable set of formulas is satisfied at infinitely many worlds. For each MCS \( w \), the worlds \( (w, 0) \) and \( (w, 1) \) will play a special role: they are forced to be irreflexive, and will be used as witnesses of \( \Diamond \)-formulas. The other copies (i.e., \( (w, n) \) for \( n \geq 2 \)) will be reflexive or irreflexive, depending on the MCS \( w \), and they may be used as witnesses of \( \bullet \)-formulas.

More precisely, we define the accessibility relation \( R \) as the set of all pairs \( ((w, n), (v, m)) \) satisfying one of the following conditions:

- \( m \leq 1 \), \( m \neq n \), and for every \( \phi \in v \), \( \Diamond \phi \in w \); and
- \( m \geq 2 \), and for every \( \phi \in v \), \( \bullet \phi \in w \).

The valuation function \( V \) for the atomic propositions is defined as usual, by letting \( V(p) = \{(w, n) \mid p \in w\} \). Let \( M = (W, R, V) \). To finish the completeness proof, we will establish the following Truth Lemma:

**Claim:** For all \( ML^* \)-formulas \( \phi \), MCSs \( w \) and natural numbers \( n, M, (w, n) \models \phi \) iff \( \phi \in w \).

This is proved by induction on \( \phi \). We will only discuss the cases where \( \phi \) is of the form \( \Diamond \psi \) or \( \bullet \psi \). First, suppose \( \phi \) is of the form \( \Diamond \psi \).

\[ \Rightarrow \] Suppose that \( M, (w, n) \models \Diamond \psi \). Then there is a \( (v, m) \) such that \( (w, n)R(v, m) \) and \( M, (v, m) \models \Diamond \psi \). By induction hypothesis, \( \psi \in v \), and by the definition of \( R \), \( \Diamond \psi \in w \) (here, we use axiom A1).

\[ \Leftarrow \] Suppose that \( \Diamond \phi \in w \). By a standard Lindenbaum construction (using K and Gen), we can obtain an MCS \( v \) containing \( \phi \) such that, for all \( \chi \in v \), \( \Diamond \chi \in w \). It follows that either \( (w, n)R(v, 0) \) or \( (w, n)R(v, 1) \), and, by induction hypothesis, the latter worlds satisfy \( \psi \). Hence, \( M, (w, n) \models \Diamond \psi \).

Next, consider \( \phi \) of the form \( \bullet \psi \).
[⇒] Suppose that $M, (w, n) \models \psi$. Then there is a pair $(v, m)$, with $m \geq 2$, such that $(w, n)R(v, m)$ and $(v, m) \models \psi$ (note that all worlds $(v, 0)$ and $(v, 1)$ are irreflexive). It follows by the definition of $R$ that $\psi \in w$.

[⇐] Suppose that $\psi \in w$.

Subclaim 1: There is an MCS $v$ containing $\psi$, such that for all $\chi \in v$, $\bullet \chi \in w$.

Proof of claim: A standard Lindenbaum construction. More concretely, let $(\chi_i)_{i \in \mathbb{N}}$ be an enumeration of all $ML^*$ formulas. The MCS $v$ is obtained as the union of an infinite chain $v_0 \subseteq v_1 \subseteq \ldots$, each satisfying $\bullet (\bigwedge \chi_i) \in w$. Let $v_0 = \{\psi\}$ and

$$v_{i+1} = \begin{cases} v_i \cup \{\chi_i\} & \text{if } \bullet ((\bigwedge \chi_i) \land \chi_i) \in w \\ v_i \cup \{-\chi_i\} & \text{otherwise} \end{cases}$$

It follows from Axiom A2 that $\bullet (\bigwedge \chi_i) \in w$ for all $i \in \mathbb{N}$. Finally, consider any $\chi \in v$. By definition of $v$, $\chi$ must belong to $v_i$ for some $i \in \mathbb{N}$. Hence $\bullet \chi \in w$ (here we use axiom A4, and the fact that $\vdash (\bigwedge \chi_i) \rightarrow \bigwedge \chi_i$). Note that, by Axiom A1, $v$ is indeed a maximal consistent set.

Subclaim 2: For all $\chi \in v$, $\diamond \chi \in v$.

Proof of claim: Suppose, for the sake of contradiction, that there is a $\chi \in v$, $\diamond \chi \notin v$. Since $v$ is an MCS, we have that $\chi \land \neg \diamond \chi \in v$. Hence, by Claim 1, $\bullet (\chi \land \neg \diamond \chi) \in w$. This contradicts Axiom A3.

It follows, for all $m \geq 2$, that $(w, n)R(v, m)$ and $(v, m) \models \psi$ (here we use the induction hypothesis), and $(v, n)$ is reflexive. Hence, $M, (w, n) \models \psi$. $\Box$

Note that $ML^*$ is not invariant for bisimulations.

C Proof of the Lindström theorem for the Binary Guarded Fragment

Before proving Theorem 3.17, we start with some modal features that hold for $GF$ in general. First, there is a natural syntactic notion of formula depth, whose inductive definition counts the above polyadic quantifiers as single steps:

- $depth(Px) = 0$
- $depth(\neg \phi) = depth(\phi)$
- $depth(\phi \lor \psi) = \max(depth(\phi), depth(\psi))$
- $depth(\exists y(G(\vec{x}, \vec{y}) \land \phi(\vec{x}, \vec{y}))) = depth(\phi) + 1$

Next, we define distance for points in models $(M, \vec{s})$, where $\vec{s}$ is a tuple of worlds:

- $dist(\vec{s}, s_i, 0)$ holds for $s_i \in \vec{s}$, and $dist(s, t, n+1)$ holds if there is a $u$ with $dist(s, u, n)$ and $G(\vec{v})$ holds for some atomic predicate $G$ and tuple of objects $\vec{v}$ containing $t$ and $u$.

We write $Cut((M, \vec{s}), n)$ for the submodel $\{t \in (M, \vec{s}) \mid dist(\vec{s}, t, n)\}$ consisting of all points $t$ in $M$ lying at distance at most $n$ from $s$. The following result shows that $GF$, like the basic modal language, satisfies a finite depth property, suitably defined:

Lemma C.1 (Distance-Depth Lemma) Let $\phi$ be any guarded formula of depth $n$, and let $(N, \vec{s})$ be any sub-model of $(M, \vec{s})$ containing all of $Cut((M, \vec{s}), n)$. Then $(M, \vec{s}) \models \phi$ iff $(N, \vec{s}) \models \phi$.

Next comes a generalization of modal bisimulation. A guarded bisimulation is a non-empty set $F$ of finite partial isomorphisms between two models $M$ and $N$ which has the following back-and-forth conditions. Call a set of objects ‘guarded’ if some tuple containing these objects stands in some atomic relation. Now, given any function $f : X \rightarrow Y$ in $F$, (i) for any guarded $Z \subseteq M$, there is a $g \in F$ with domain $Z$ such that $g$ and $f$ agree on the intersection $X \cap Z$, (ii) for any guarded $W \subseteq N$, there is a $g \in F$ with range $W$ such that the inverses $g^{-1}$ and $f^{-1}$ agree on $Y \cap W$.

Also, ‘rooted’ guarded bisimulations $F$ run between models $(M, \vec{s})$ and $(N, \vec{t})$ with given initial objects, where one requires that some match between $\vec{s}, \vec{t}$ is already a partial isomorphism in $F$. By a simple inductive argument, $GF$-formulas are invariant for rooted guarded bisimulations.

Andréka, van Benthem and Németi 1998 show that $GF$ consists, up to logical equivalence, of just those first-order formulas which are invariant for guarded bisimulations. Another ‘modal’ use of guarded bisimulation in the same paper is model unraveling. This is like standard modal unraveling, but the construction is a bit more delicate:

Definition C.3 (Tree unravelings for $GF$) The tree unraveling $Unr(M)$ of $M$ has for its objects all pairs $(\pi, d)$ where the ‘path’ $\pi$ is a finite sequence of guarded sets in $M$, and the M-object $d$ is ‘new’ in $\pi$: it occurs in the final set of $\pi$ but not in the one before that. The interpretation of predicate symbols $Q$ on these objects $(\pi, d)$ is as follows. $I(Q)$ holds for a finite sequence of objects $((\pi_1, d_1), \ldots, (\pi_k, d_k))$ iff $Q^M(d_1, \ldots, d_k)$ and there is some maximal path $\pi^*$ among those listed of which all other $\pi_i$ are initial segments in such a way that their new objects $d_i$ remain present in each set until the end of $\pi^*$. For a model $(M, \vec{s})$ this generalizes as follows. $Unr(M, \vec{s})$ has paths $\pi$ all starting from the initial set $\vec{s}$, but then continuing with guarded sets only. The objects $(\pi, d)$ are defined as before.

The point here is that the set $F$ of all restrictions of the finite maps sending $(\pi_1, d_1)$ to $d_1$ for all guarded finite domains in $Unr(M, \vec{s})$ is a rooted guarded bisimulation between $(M, \vec{s})$ and $(Unr(M), \vec{s})$. Checking the zigzag conditions for the bisimulation will reveal the reason for the above definition of the predicate interpretations $I(Q)$. 

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Now we have the generalities in place for our Lindström Theorem, but it remains to make some adjustments. First, all these general \(GF\) notions and observations specialize to \(GFbin\) in an obvious way, which we do not bother to spell out. Less generally, in the definition of tree unravelings, we make one simple change for the binary case:

The finite paths of guarded sets always introduce one new object at each stage. At each step, one takes a new object related to that new object.

This allows paths starting with object \(a\) and then continuing with \(Rab, Qch, \ldots\), while ruling out paths like \(Rab, Qac\). But the final atom is not omitted from the unraveled model, since one can have paths starting with \(a\) and then placing \(Qac\) immediately. Thus, even with these restricted paths, we still have a guarded bisimulation between tree unravelings and their original models. The real point of this adjustment is the following.

The definition of predicates for path objects now makes binary relations hold only between objects \((\pi_1, d_1), (\pi_2, d_2)\) where \(\pi_2\) is a one-step continuation of the path \(\pi_1\), or vice versa. But then, counting distance as before,

The new object at the end of a path of length \(k\) lies at distance \(k\) from the initial object of the path.

Put in more vivid terms, ‘tree distance is true distance’ in the original model. This is a non-obvious fact. E.g., with ternary guards \(Rayz\), objects at the end of a path may keep links to the initial object a which might recur in the guarded sets along the path (this observation is due to Martin Otto).

**Proof of Theorem 3.17:** Let \(L\) be any abstract logic extending \(GF\) and satisfying Compactness and invariance for guarded bisimulations. As before, it suffices to show that every formula \(\phi \in L\) is invariant for models that are equivalent for all \(GFbin\)-formulas up to some finite depth \(n\).

First, as in the earlier modal proof of Sect. 3.1, we use the Compactness of \(L\), together with its Relativization closure, to show that \(\phi\) must have the Finite Occurrence Property and a Finite Distance Property for some level \(n\). Before, universal prefix formulas \(\square^k p\) (for all finite \(k\)) made sure that \(p\) holds in the generated submodel at the current world. This time, one uses all nested sequences of universal guarded quantifiers up to depth \(k\), requiring that some new predicate \(P\) holds for all objects reached at the end. The \(n\) thus found for the local depth of the formula \(\phi\) is the same \(n\) as needed for the following semantic invariance:

Given the above unraveling construction and Invariance for Guarded Bisimulation for \(L\), we may assume, without loss of generality, that we have the following situation:

(a) \(\langle Unr(M), \bar{s} \rangle \models \phi\)

(b) \(\langle Unr(M), \bar{s} \rangle\) is \(GFbin\)-\(n\)-equivalent to \(\langle Unr(N), \bar{t} \rangle\)

Our aim is to show that \(\langle Unr(N), \bar{t} \rangle \models \phi\).

We cut the tree models to depth \(n\), as before in our modal argument, obtaining \(Cut((Unr(M), \bar{s}), n)\), \(Cut((Unr(N), \bar{t}), n)\). Since tree depth is true depth, this does not change truth values of \(\phi\) in either model.

Next, we have as usual that there is an \(n\)-tower of partial isomorphisms \(PI_n, \ldots, PI_0\), starting from the link \(\bar{s}, \bar{t}\), which satisfies the guarded back and forth properties. Stage \(j\) contains tuples (in fact, guarded pairs) of objects satisfying the same \(GFbin\) formulas up to syntactic depth \(n - j\). The back and forth properties are proved by using, at stage \(j + 1\), guarded existential quantifiers describing the next object to be linked up to syntactic level \(j\). In particular, working in our tree models, we can make sure that

(\#) The finite partial isomorphisms at level \(PI_j\) of the tower \((j < n)\) are between guarded sets of two objects lying at distances \(n - k, n - k - 1\) from the root objects, which satisfy the same formulas of \(GFbin\)-\(k\).

Our crucial claim, as in the basic modal case, is that

The union of all \(PI_j\) is a \(GFbin\)-bisimulation.

The only thing to be checked here is that the partial isomorphisms in \(PI_0\) still satisfy the guarded back and forth properties. In the purely modal case, this was because end points of the cut-off tree have no successors, and hence there is nothing to be proved. In the present case, points at distance \(n\) may have more significant relationships with objects in the cut-off tree, but given our definition of atomic predicates for path objects, these can only be of special forms. If the objects \(u, v\) in the guarded pair lie at distance \(< n\) from a root object, we are done by (\#). And if, say, \(u\) lies at distance \(n\) from the root, the only significant binary relationship it can have in our cut-off tree model is with its companion object \(v\) – which was unique, by our path construction. But then, the current partial isomorphism itself provides the required back-and-forth match.

Finally, by invariance for guarded bisimulations, the truth of \(\phi\) transfers from \(Cut((Unr(M), \bar{s}), n)\) to \(Cut((Unr(N), \bar{t}), n)\), and hence to \(\langle N, \bar{t} \rangle\). \(\square\)