INTERPOLATION FOR EXTENDED MODAL LANGUAGES.
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Abstract. Several extensions of the basic modal language are characterized in terms of interpolation. Our main results are of the following form: *Language $\mathcal{L}'$ is the least expressive extension of $\mathcal{L}$ with interpolation.* For instance, let $\mathcal{M}(D)$ be the extension of the basic modal language with a difference operator [8]. First-order logic is the least expressive extension of $\mathcal{M}(D)$ with interpolation. These characterizations are subsequently used to derive new results about difference logic, hybrid logic, relation algebra and the guarded fragment.

§1. Introduction. In this paper, we consider extensions of the basic modal language that involve reference to individual states of a Kripke structure. A typical example is the language $\mathcal{H}(E)$, in which one can refer to individual states of the Kripke model using nominals (similar to constants in first-order logic) and the universal modality [10]. Another example is difference logic $\mathcal{M}(D)$, i.e., the extension of the basic modal language with an extra operator $D$ such that $\mathcal{M}(D)$ and $\mathcal{H}(E)$ are known not to have interpolation [5]. In this paper, we systematically investigate extensions of these languages, trying to restore interpolation. We show that, in a precise sense, first-order logic is the smallest extension of these languages with interpolation. This can be seen as a characterization of first-order logic or, from another perspective, as a general negative interpolation result. In a similar way, we characterize the hybrid language $\mathcal{H}(\#, \uparrow)$ as the smallest extension of $\mathcal{H}(\#)$ with interpolation [2].

These characterizations are subsequently used to derive new results concerning interpolation for hybrid logic, difference logic, relation algebra and the guarded fragment. In the case of hybrid logic, we prove as a corollary that there is no decidable hybrid language with Craig interpolation, thus answering an open question raised by Areces, Blackburn, and Marx [2].

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Section 2 provides an abstract definition of modal languages that is general enough to cover also languages involving nominals, such as $\mathcal{H}(E)$ and $\mathcal{H}(\emptyset)$.

Section 3 introduces a number of specific languages. Section 4 contains the main results, namely the characterizations of first-order logic and $\mathcal{H}(\emptyset, 1)$ mentioned above. Section 5 presents applications of these results.

§2. Abstract Modal Languages. Throughout Section 2–4, we will assume a given, fixed set of modalities $\Omega$, and a variable signature $\sigma = (\text{PROP}_\sigma, \text{NOM}_\sigma)$ consisting of disjoint sets containing proposition letters and nominals respectively. The nominals will act like proposition letters, except that their denotation in a model will always be a singleton set (i.e., they denote points rather than sets). We will often be sloppy by using $\sigma$ to denote the union $\text{PROP}_\sigma \cup \text{NOM}_\sigma$. Each modality $\Delta \in \Omega$ has an associated arity $n(\Delta)$.

A frame is a structure $\mathfrak{F} = (W, R_\Delta)_{\Delta \in \Omega}$, where $W$ is a non-empty set of worlds and $R_\Delta \subseteq W^{n(\Delta)+1}$ for each $\Delta \in \Omega$. A (pointed) $\sigma$-model is a structure $\mathcal{M} = (\mathfrak{F}, V, w)$ where $\mathfrak{F} = (W, R_\Delta)_{\Delta \in \Omega}$ is a frame, $V : \text{PROP}_\sigma \cup \text{NOM}_\sigma \rightarrow \wp(W)$ and $w \in W$, such that $V$ assigns singleton sets to the elements of $\text{NOM}_\sigma$. $\text{Str}[\sigma]$ denotes the class of all $\sigma$-models. Furthermore, for any class of frames $\mathcal{F}$, $\text{Str}[\sigma]$ denotes the class of all $\sigma$-models of which the underlying frame belongs to $\mathcal{F}$.

Two operations on models will be useful later on. Firstly, let a renaming $\rho : \sigma \rightarrow \tau$ be a mapping from $\sigma$ to $\tau$ that respects the sorting: it maps elements of $\text{PROP}_\sigma$ to elements of $\text{PROP}_\tau$ and elements of $\text{NOM}_\sigma$ to elements of $\text{NOM}_\tau$. For any model $\mathcal{M} = (\mathfrak{F}, V, w) \in \text{Str}[\tau]$ and renaming $\rho : \sigma \rightarrow \tau$, let $\mathcal{M}^\rho$ be the $\sigma$-model $(\mathfrak{F}, \rho \cdot V, w)$. Secondly, if $\mathcal{M} \in \text{Str}[\tau]$ and $\sigma \subseteq \tau$, then $\mathcal{M} \models \sigma$ denotes the $\sigma$-reduct of $\mathcal{M}$, i.e., the $\sigma$-model that is obtained from $\mathcal{M}$ by “forgetting” the interpretation of $\tau \setminus \sigma$. We write $K \models \sigma$ for $\{\mathcal{M} \mid \sigma \models K\}$.

Definition 2.1 (Modal languages). A modal language is a pair $(\mathcal{L}, \models_{\mathcal{L}})$, where $\mathcal{L}$ is a map from signatures to sets of formulas, and $\models_{\mathcal{L}}$ is a relation between formulas and models satisfying the following conditions.

1. Expansion Property. If $\sigma \subseteq \tau$ then $\mathcal{L}[\sigma] \subseteq \mathcal{L}[\tau]$. For all $\phi \in \mathcal{L}[\sigma]$ and $\mathcal{M}, w \in \text{Str}[\sigma \cup \tau]$, $\mathcal{M}, w \models_{\mathcal{L}} \phi$ iff $\mathcal{M} \models_{\mathcal{L}} [\sigma \cup \tau] w \models_{\mathcal{L}} \phi$. If $\mathcal{M} \in \text{Str}[\sigma]$ and $\mathcal{M}, w \models_{\mathcal{L}} \phi$, then $\phi \in \mathcal{L}[\sigma]$.

2. Renaming Property For all $\phi \in \mathcal{L}[\sigma]$ and renamings $\rho : \sigma \rightarrow \tau$, there is a $\psi \in \mathcal{L}[\tau]$ such that for all $\mathcal{M} \in \text{Str}[\tau]$, $\mathcal{M} \models_{\mathcal{L}} \psi$ iff $\mathcal{M}^\rho \models_{\mathcal{L}} \phi$.

Definition 2.1 is inspired by similar ones occurring in the literature on abstract model theory [3]. Since the definition is rather general, one might ask what is still modal about modal languages according to this definition. The two main distinctively modal features in the definition are (1) the fact that the structures we work with are pointed, reflecting the fact that modal formulas are always evaluated locally, and (2) the strict distinction between modalities on the one hand and proposition letters and nominals on the other hand. The importance of this distinction will become clear later on, when we’ll consider specific classes of frames.

The typical example of a modal language is the basic modal language $\mathcal{BML}$ [4]. This particular language doesn’t make use of nominals: for any signature $\sigma = (\text{PROP}, \text{NOM}), \mathcal{BML}[\sigma] = \mathcal{BML}[(\text{PROP}, \emptyset)]$. In fact, nominals are included in
the signature only for convenience, since this makes it possible to classify various hybrid languages simply as extensions of the basic modal language. Section 3 contains more examples of modal languages.

Some shorthand notation will be convenient. Firstly, par abus de langage, we will often use $\mathcal{L}$ to refer to the pair $(\mathfrak{F}, \models)$. Secondly, given a model $\mathfrak{M} = (\mathfrak{F}, V, w)$ and a point $v \in W$, we will use $(\mathfrak{M}, v)$ to denote the model $(\mathfrak{F}, V, v)$. Thus, with $\mathfrak{M}, v \models \phi$ we mean $(\mathfrak{F}, V, v) \models \phi$. Next, for $\phi \in \mathcal{L}^{[\sigma]}$, let $\text{Mod}^{\sigma}_\mathcal{L}(\phi) = \{\mathfrak{M} \in \text{Str}^{[\sigma]} \mid \mathfrak{M} \models_\mathcal{L} \phi\}$. Finally, the symbol $\models$ will be used not only to refer to the satisfaction relation, but also to the local consequence relation: for $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{[\sigma]}$, we say that $\Phi \models_\mathcal{L} \phi$ if $\bigcap_{\phi \in \Phi} \text{Mod}^{\sigma}_\mathcal{L}(\phi) \subseteq \text{Mod}^{\sigma}_\mathcal{L}(\psi)$.

Often, we will restrict attention to a specific frame class $\mathcal{F}$. In these cases, we will write $\text{Mod}^{\sigma}_\mathcal{F}(\phi)$ for $\{\mathfrak{M} \in \text{Str}^{[\sigma]} \mid \mathfrak{M} \models_\mathcal{L} \phi\}$. Likewise, for $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{[\sigma]}$, we say that $\Phi \models_{\mathcal{F}, \mathcal{L}} \phi$ if $\bigcap_{\phi \in \Phi} \text{Mod}^{\sigma}_{\mathcal{F}, \mathcal{L}}(\phi) \subseteq \text{Mod}^{\sigma}_{\mathcal{F}, \mathcal{L}}(\psi)$.

**Definition 2.2** (Extensions of modal languages). $\mathcal{L}'$ extends $\mathcal{L}$ relative to a frame class $\mathcal{F}$ (notation: $\mathcal{L} \leq_{\mathcal{F}} \mathcal{L}'$) if the following holds for all signatures $\sigma$ and proposition letters $p_1, \ldots, p_n$ ($n \geq 0$).

- For each $\phi \in \mathcal{L}^{[\sigma]} \cup \{p_1, \ldots, p_n\}$ and $\psi_1, \ldots, \psi_n \in \mathcal{L}^{[\sigma]}$, there is a formula of $\mathcal{L}'^{[\sigma]}$, which we will denote by $\phi[\psi_1/\psi_n]$, such that $\text{Mod}^{\sigma}_{\mathcal{F}, \mathcal{L}'}(\phi[\psi_1/\psi_n]) = \{\mathfrak{M} \in \text{Str}^{[\sigma]} \mid \mathfrak{M} \models_\mathcal{F} \phi, \mathfrak{M} \models_\mathcal{L} \psi_1, \ldots, \mathfrak{M} \models_\mathcal{L} \psi_n\} \models_\mathcal{L} \phi$.

Note that Definition 2.2 concerns expressive extensions rather than axiomatic extensions. As a special case (take $n = 0$), we have that whenever $\mathcal{L} \leq_{\mathcal{F}} \mathcal{L}'$ and $\phi \in \mathcal{L}^{[\sigma]}$, there is a $\psi \in \mathcal{L}^{[\sigma]}$ such that $\text{Mod}^{\sigma}_{\mathcal{F}, \mathcal{L}'}(\phi) = \text{Mod}^{\sigma}_{\mathcal{F}, \mathcal{L}'}(\psi)$. However, Definition 2.2 provides more information: it ensures that $\mathcal{L}'$ is closed under the basic operations of $\mathcal{L}$, such as negation. Definitions like Definition 2.2 are quite common in the literature on abstract model theory. Incidentally, it should be noted that the definition makes sense only for languages for languages $\mathcal{L}$ closed under substitution, in the sense that otherwise it might happen that $\mathcal{L} \not\subseteq \mathcal{L}$.

Finally, let us define interpolation. In the context of modal logic, interpolation can be defined in several ways. The following definition captures what is often called local interpolation or arrow interpolation (keep in mind that $\models_\mathcal{L}$ is the local consequence relation).

**Definition 2.3** (Interpolation). A modal language $\mathcal{L}$ has interpolation on a frame class $\mathcal{F}$ if for all $\phi \in \mathcal{L}^{[\sigma]}$ and $\psi \in \mathcal{L}^{[\tau]}$ such that $\phi \models_{\mathcal{F}, \mathcal{L}} \psi$, there is a $\theta \in \mathcal{L}^{[\sigma \cap \tau]}$ such that $\phi \models_{\mathcal{F}, \mathcal{L}} \theta$, and $\theta \models_{\mathcal{F}, \mathcal{L}} \psi$.

Note that, since the modalities are not part of the signature, it is not required that the interpolant $\theta$ only contains modalities occurring both in $\phi$ and in $\psi$.

**§3. Examples of modal languages.** As was already mentioned, the basic modal language is a modal language in the sense of Definition 2.1. In this section, we introduce a number of other modal languages. Recall from the previous section that we assume a given set of modalities $\Omega$, and each element $\Delta \in \Omega$ has an associated arity $n(\Delta)$.

- The language of difference logic, which we will denote by $\mathcal{M}(D)$, is the extension of the basic modal language with an extra operator $D$, such that
\( \mathcal{M}, w \models \phi \) iff \( \mathcal{M}, v \models \phi \) for some world \( v \neq w \) \[8\]. As was the case for the basic modal language, nominals remain unused.

- The basic hybrid language \( \mathcal{H}(\@) \) is obtained by extending the basic modal language with nominals \( i, j, k, \ldots \) and with a satisfaction operator \( @i \) for each nominal \( i \). For example, the formula \( \Box \Diamond i \) holds at a point \( w \) just in case every successor of \( w \) can see the unique point at which the nominal \( i \) is true, and \( @i \phi \) holds just in case \( \phi \) is true at the point named by the nominal \( i \). More precisely, given a signature \( \sigma = (\text{PROP}, \text{NOM}) \), the formulas of \( \mathcal{H}(\@)[\sigma] \) are given by \( \phi := p \mid i \mid \top \mid \phi \land \psi \mid \neg \phi \mid \Delta(\phi_1, \ldots, \phi_{n(\Delta)}) \mid @i \phi \) where \( p \in \text{PROP} \), \( i \in \text{NOM} \) and \( \Delta \in \Omega \). For any model \( \mathcal{M} = (W, R, V, w) \in \text{Str}[\sigma] \), \( \mathcal{M} \models i \) iff \( V(i) = \{w\} \) and \( \mathcal{M} \models @i \phi \) iff \( \mathcal{M}, v \models \phi \) where \( V(i) = \{v\} \).

- The slightly more expressive hybrid language \( \mathcal{H}(\mathcal{E}) \) is obtained by extending the basic modal language with nominals and the global modality \( \mathcal{E} \). Formally, the formulas of \( \mathcal{H}(\mathcal{E}) \) are given by \( \phi := p \mid i \mid \neg \phi \mid \phi \land \psi \mid \Delta(\phi_1, \ldots, \phi_{n(\Delta)}) \mid \mathcal{E} \phi \), where \( p \in \text{PROP} \), \( i \in \text{NOM} \) and \( \Delta \in \Omega \). The truth definition is such that \( \mathcal{M}, w \models \mathcal{E} \phi \) iff \( \mathcal{M}, v \models \phi \) for some world \( w \) of \( \mathcal{M} \). Notice that the satisfaction operators are definable in terms of the global modality: \( @i \phi \) is equivalent to \( \mathcal{E}(i \land \phi) \).

- Another very expressive hybrid language \( \mathcal{H}(\@, \downarrow) \) is obtained by extending \( \mathcal{H}(\@) \) with the \( \downarrow \)-binder, which allows explicit reference to the current point of evaluation. For example, the formula \( \downarrow.x \Diamond x \) expresses that the current world is a successor of itself. Formally, let \( \text{VAR} \) be a countably infinite set of variables, and for any formula \( \phi \), nominal \( i \) and variable \( x \), let \( \phi[i/x] \) be the result of replacing all occurrences of \( i \) in \( \phi \) by \( x \). Then the formulas of \( \mathcal{H}(\@, \downarrow) \) are given by \( \phi := p \mid i \mid \neg \phi \mid \phi \land \psi \mid \Delta(\phi_1, \ldots, \phi_{n(\Delta)}) \mid @i \phi \mid \downarrow.x \phi[i/x] \), where \( p \in \text{PROP} \), \( i \in \text{NOM} \), \( \Delta \in \Omega \) and \( x \in \text{VAR} \), such that \( x \) is substitutable for \( i \) in \( \phi \). Note that this definition uses a trick familiar from first-order logic to avoid free variables. The truth definition is such that \( \mathcal{M}, w \models \downarrow.x \phi[i/x] \) iff \( \mathcal{M}^{[i=(w)]}, w \models \phi \), where the model \( \mathcal{M}^{[i=(w)]} \) is identical to \( \mathcal{M} \) except that \( i \) denotes \( \{w\} \).

- First-order logic also constitutes a modal language, in the following sense. For every signature \( \sigma = (\text{PROP}, \text{NOM}) \), let \( \sigma^* \) be the first-order logic signature that has \( \text{PROP} \) as its unary predicates, \( \text{NOM} \) as its constants, and that has a relation \( R_\Delta \) of arity \( n(\Delta) + 1 \) for each \( \Delta \in \Omega \). Fix a first-order variable \( x \), and for all signatures \( \sigma \), let \( \mathcal{L}^1[\sigma] \) be collection of first-order formulas with no free variables besides \( x \), in the signature \( \sigma^* \) with equality. Furthermore, let \( \mathcal{M}, w \models \mathcal{L}_1^1 \phi(x) \) if \( \phi(x) \) holds in \( \mathcal{M} \) conceived of as an ordinary first-order structure, interpreting \( x \) as \( w \) and \( R_\Delta \) as the accessibility relation for \( \Delta \). Then \( (\mathcal{L}_1^1, \models_{\mathcal{L}_1^1}) \) is a modal language, and we will refer to it as the first-order correspondence language, or simply first-order logic.

- Finally, we define \( \mathcal{L}_1^{-} \) to be the first-order correspondence language without nominals (i.e., without first-order constants). In other words, for any signature \( \sigma = (\text{PROP}, \text{NOM}) \), \( \mathcal{L}_1^{-}[\sigma] = \mathcal{L}_1^1[(\text{PROP}, \emptyset)] \).

Of course, there are many other modal languages, including infinitary and higher-order languages. However, the above mentioned ones will play an important role in the next section. Definition 2.2 orders these modal languages into the
hierarchy depicted in Figure 1. Of the languages depicted here, the basic modal language is the least expressive and the first-order correspondence language \( L^1 \) is the most expressive. Assuming we have at least one unary modality and that we work with the class of all frames, all inclusions are strict. \( \ominus \) indicates that a language has interpolation on the class of all frames, \( \ominus \) indicates that it does not. All these results are known, cf. \[8, 10, 2, 6, 9, 5\].

§4. Main results. The following results, which form the main contribution of this paper, complement the known results depicted in Figure 1. They show that every language between \( \mathcal{H}(\otimes) \) and \( \mathcal{H}(\otimes, \bot) \), between \( \mathcal{H}(E) \) and \( L^1 \) or between \( \mathcal{M}(D) \) and \( L^1 \) lacks interpolation. In fact, this holds relative to any frame class \( F \).

Theorem 4.1. Let \( \mathcal{L} \) be any modal language, and let \( F \) be any class of frames (we make no assumptions on the collection of modalities \( \Omega \)). The following hold.

1. If \( \mathcal{H}(\otimes) \subseteq_F \mathcal{L} \) and \( \mathcal{L} \) has interpolation on \( F \) then \( \mathcal{H}(\otimes, \bot) \subseteq_F \mathcal{L} \)
2. If \( \mathcal{H}(E) \subseteq_F \mathcal{L} \) and \( \mathcal{L} \) has interpolation on \( F \) then \( L^1 \subseteq_F \mathcal{L} \)
3. If \( \mathcal{M}(D) \subseteq_F \mathcal{L} \) and \( \mathcal{L} \) has interpolation on \( F \) then \( L^{1-} \subseteq_F \mathcal{L} \)

These results can be interpreted as general negative interpolation results, or, from another perspective, as characterizations. For instance, Theorem 4.1.1 characterizes \( \mathcal{H}(\otimes, \bot) \) as the smallest extension of \( \mathcal{H}(\otimes) \) that has interpolation. Incidentally, for those readers familiar with abstract model theory, it should be mentioned that something slightly stronger holds: \( \mathcal{H}(\otimes, \bot) \) is the \( \Delta \)-closure of \( \mathcal{H}(\otimes) \), \( L^1 \) is the \( \Delta \)-closure of \( \mathcal{H}(E) \) and \( L^{1-} \) is the \( \Delta \)-closure of \( \mathcal{M}(D) \).

The remainder of this section is devoted to the proof of Theorem 4.1. First, we prove an adapted version of well-known lemma relating interpolation with projective classes [3].

Definition 4.1 (Projective classes). \( K \subseteq \text{Str}_F[\sigma] \) is a projective class of \( \mathcal{L} \) relative to \( F \) if there is a \( \phi \in \mathcal{L}[\tau] \) with \( \tau \supseteq \sigma \) such that \( K = \text{Mod}_{\mathcal{L},F}(\phi) \upharpoonright \sigma \).

Definition 4.2. A modal language \( \mathcal{L} \) has negation on \( F \) if for each \( \phi \in \mathcal{L}[\sigma] \) there is an formula of \( \mathcal{L}[\sigma] \), which we will denote by \( \neg \phi \), such that \( \text{Mod}_{\mathcal{L},F}(\psi) = \text{Str}_F[\sigma] \setminus \text{Mod}_{\mathcal{L},F}(\phi) \).

Lemma 4.1. Let \( \mathcal{L} \) be a modal language with negation that has interpolation on \( F \), and let \( K \subseteq \text{Str}_F[\sigma] \). If both \( K \) and \( \text{Str}_F[\sigma] \setminus K \) are projective classes of \( \mathcal{L} \) relative to \( F \), then there is a \( \phi \in \mathcal{L}[\sigma] \) such that \( K = \text{Mod}_{\mathcal{L},F}(\phi) \).
PROOF. Since \( K \) is a projective class, there is a formula \( \phi \in L[\sigma \cup \tau] \) such that \( K = \text{Mod}_{L_{SF}}(\phi) \setminus \sigma \). Likewise, since \( \text{Str}_F[\sigma] \setminus K \) is a projective class, there is a formula \( \psi \in L[\sigma \cup \tau'] \) such that \( \text{Str}_F[\sigma] \setminus K = \text{Mod}_{L_{SF}}(\psi) \setminus \sigma \). Without loss of generality, we can assume that \( \tau \) and \( \tau' \) are disjoint (by the Renaming property of \( L \)). It follows that \( \phi \models_{L,F} \neg \psi \). Since \( L \) has interpolation, there must be a \( \theta \in L[\sigma] \) such that \( \phi \models_{L,F} \theta \) and \( \theta \models_{L,F} \neg \psi \). As a last step, we will show that \( \text{Mod}_{L_{SF}}(\theta) = K \).

Suppose \( \mathfrak{M} \in K \). Then \( \mathfrak{M} = \mathfrak{N} \models \sigma \) for some \( \mathfrak{N} \in \text{Mod}_{L_{SF}}(\phi) \). Since \( \phi \models_{L,F} \theta \), it follows that \( \mathfrak{N} \models \theta \). By the Expansion property, \( \mathfrak{M} \models \theta \). Conversely, suppose \( \mathfrak{M} \not\models \theta \). Then \( \mathfrak{M} = \mathfrak{N} \models \sigma \) for some \( \mathfrak{N} \in \text{Mod}_{L_{SF}}(\psi) \). Since \( \theta \models_{L,F} \neg \psi \), it follows that \( \mathfrak{N} \models \neg \theta \). By the Expansion property, \( \mathfrak{M} \models \neg \theta \).

Using Lemma 4.1, we can show that whenever the \( \downarrow \)-binder is added to a modal language that has interpolation and extends either \( H(\emptyset) \) or \( M(D) \), the expressivity of the language in question does not increase. This is expressed in the following two Lemmas.

**Lemma 4.2.** Let \( L \) be a modal language with interpolation on \( F \) such that \( H(\emptyset) \subseteq_F L \). For all \( \phi \in L[\sigma] \) and \( i \in \text{NOM}_\sigma \), there is a formula of \( L[\sigma \setminus \{i\}] \), which we will denote by \( \downarrow \phi \), such that \( \text{Mod}_{L_{SF}}(\downarrow \phi) = \{ (\mathfrak{N}, V, w) \in \text{Str}_F[\sigma \setminus \{i\}] \mid (\mathfrak{N}, V^{\downarrow \phi}, w) \models \phi \} \).

**Proof.** Let \( K_{\downarrow \phi} = \{ (\mathfrak{N}, V, w) \in \text{Str}_F[\sigma \setminus \{i\}] \mid (\mathfrak{N}, V^{\downarrow \phi}, w) \models \phi \} \). \( K_{\downarrow \phi} \) is projectively defined by \( \downarrow \phi \) and its complement is projectively defined by \( \downarrow \neg \phi \). Since \( L \) has interpolation and has interpolation on \( F \), by Lemma 4.1 \( K_{\downarrow \phi} = \text{Mod}_{L_{SF}}(\psi) \) for some \( \psi \in L[\sigma \setminus \{i\}] \).

**Lemma 4.3.** Let \( L \) be a modal language with interpolation on \( F \) such that \( M(D) \subseteq_F L \). For all \( \phi \in L[\sigma] \) and \( p \in \text{PROP}_\sigma \), there is a formula of \( L[\sigma \setminus \{p\}] \), which we shall denote by \( \downarrow p \phi \), such that \( \text{Mod}_{L_{SF}}(\downarrow p \phi) = \{ (\mathfrak{N}, V, w) \in \text{Str}_F[\sigma \setminus \{p\}] \mid (\mathfrak{N}, V^{\downarrow p \phi}, w) \models \phi \} \).

**Proof.** Let \( K_{\downarrow p \phi} = \{ (\mathfrak{N}, V, w) \in \text{Str}_F[\sigma \setminus \{p\}] \mid (\mathfrak{N}, V^{\downarrow p \phi}, w) \models \phi \} \). \( K_{\downarrow p \phi} \) is projectively defined by \( \downarrow p \phi \) and its complement is projectively defined by \( \downarrow \neg p \phi \). Since \( L \) has interpolation and has interpolation on \( F \), by Lemma 4.1 \( K_{\downarrow p \phi} = \text{Mod}_{L_{SF}}(\psi) \) for some \( \psi \in L[\sigma \setminus \{p\}] \).

We are now ready to prove Theorems 4.1.1–4.1.3.

**Proof of Theorem 4.1.1.** Let \( L \) be any modal language with interpolation on \( F \) such that \( H(\emptyset) \subseteq_F L \) and let \( \phi \in H(\emptyset, \downarrow)[\sigma] \). We will show that there is a formula \( \chi \in L[\sigma] \) that is equivalent to \( \phi \) on \( F \). The proof proceeds by induction on the length of \( \phi \). The base case (where \( \phi \) is a proposition letter or nominal or \( \top \)) follows immediately from the fact that \( H(\emptyset) \subseteq_F L \). For the inductive step, we will only prove the cases for negation and for the \( \downarrow \)-binder, since the other cases are similar to the one for negation.

Let \( \phi \) be of the form \( \neg \psi \). By induction hypothesis, \( \psi \) is equivalent on \( F \) to some \( \chi \in L[\sigma] \). Let \( p \) be any proposition letter not in \( \sigma \). Since \( H(\emptyset) \subseteq_F L \) and \( \neg \phi \in H(\emptyset)[\sigma \cup \{p\}] \), Definition 2.2 guarantees the existence of a formula \( (\neg \phi)^p \models \chi \) that is equivalent to \( \phi \) on \( F \).
Let $\phi$ be of the form $\langle x, \psi \rangle$. Let $i$ be any nominal not in $\sigma$. By the induction hypothesis, we know that there is some $\chi \in \mathcal{L}[\sigma \cup \{i\}]$ that is equivalent on $F$ to $\psi[x/i]$. By Lemma 4.2 it follows that $\langle x, \psi \rangle$ is equivalent on $F$ to $\langle i, \chi \rangle \in \mathcal{L}[\sigma]$. $\dashv$

**Proof of Theorem 4.1.2.** Similar to the proof of Theorem 4.1.1. We will only discuss the inductive step for formulas of the form $\exists y. \psi$. Let $\phi \in \mathcal{L}^1[\sigma]$ be of the form $\exists y. \psi$. By the definition of $\mathcal{L}^1$, $\phi$ contains at most one free variable, say $x$ (in case $\phi$ contains no free variables, let $x$ be any variable distinct from $y$). Let $i, j$ be distinct nominals (constants) not in $\sigma$. By induction hypothesis, $\phi[x/i, y/j] \in \mathcal{L}^1[\sigma \cup \{i, j\}]$ is equivalent on $F$ to some $\chi \in \mathcal{L}[\sigma \cup \{i, j\}]$. By Lemma 4.2 and by the fact that $\mathcal{H}(E) \subseteq \mathcal{L}$, we obtain a formula $\langle i, \mathcal{E} \rangle \langle j, \chi \rangle \in \mathcal{L}[\sigma]$ that is easily shown to be equivalent to $\phi$ on $F$. $\dashv$

**Proof of Theorem 4.1.3.** To simplify the induction, we will temporarily extend the syntax of $\mathcal{L}^1$, by allowing unary predicates to occur as arguments of other predicates. For instance, $R(y, P)$ is now allowed as a formula of $\mathcal{L}^1$ and it is interpreted as $\exists x (P x \land R y x)$. This change clearly does not affect the expressive power of $\mathcal{L}^1$, but it will make the inductive argument simpler. The proof now proceeds similarly to that of Theorem 4.1.1 and 4.1.2 (note that the universal modality is definable in terms of the difference operator: $\mathcal{E} \phi$ is equivalent to $\phi \lor \mathcal{D} \phi$). We will only provide the inductive argument for formulas of the form $\exists y. \psi$.

Let $\phi \in \mathcal{L}^1[\sigma]$ be of the form $\exists y. \psi$. By the definition of $\mathcal{L}^1$, $\phi$ contains at most one free variable, say $x$ (in case $\phi$ contains no free variables, let $x$ be any variable distinct from $y$). Let $p, q$ be distinct proposition letters not in $\sigma$. By induction hypothesis, $\phi[x/p, y/q] \in \mathcal{L}^1[\sigma \cup \{p, q\}]$ is equivalent on $F$ to some $\chi \in \mathcal{L}[\sigma \cup \{p, q\}]$. By Lemma 4.3 and the fact that $\mathcal{M}(D) \subseteq \mathcal{L}$, we can obtain a formula $\langle p, \mathcal{E} \rangle \langle q, \chi \rangle \in \mathcal{L}[\sigma]$ that is easily shown to be equivalent to $\phi$ on $F$. $\dashv$

§5. Applications.

5.1. Axiomatic extensions of $\mathcal{K}_{\mathcal{M}(D)}$ and $\mathcal{K}_{\mathcal{H}(E)}$. Let $\mathcal{M}_4(D)$ be the temporal version of $\mathcal{M}(D)$, i.e., the extension of $\mathcal{M}(D)$ with the converse modalities (for each $n$-ary modality $\Delta \in \Omega$, $\mathcal{M}_4(D)$ contains all converse modalities). Then as a corollary of Theorems 4.1.2 and 4.1.3, we obtain the following negative interpolation result for $\mathcal{M}(D)$ and $\mathcal{H}(E)$.

**Proposition 5.1.** Let $F$ be any non-empty frame class closed under disjoint union. Then $\mathcal{M}(D)$, $\mathcal{M}_4(D)$ and $\mathcal{H}(E)$ lack interpolation on $F$.

**Proof.** By Theorems 4.1.2 and 4.1.3, it suffices to show that $\mathcal{M}_4(D)$ and $\mathcal{H}(E)$ lack full first-order expressivity on $F$ (note that $\mathcal{M}(D) \subseteq \mathcal{F} \mathcal{M}_4(D)$). In fact, we will show that $\exists x y z (P x \land P y \land P z \land x \neq y \land x \neq z \land y \neq z)$ is not expressible. Take any frame $\mathfrak{F} \in F$, and let $w \in \mathfrak{F}$. Let $\mathfrak{G}$ be the disjoint union of five copies of $\mathfrak{F}$, and let $w_1, \ldots, w_5$ be the five copies of $w$ in $\mathfrak{G}$. As $F$ is closed under disjoint union, $\mathfrak{G} \in F$. Consider the signature that consists of precisely one proposition letter $p$ and no nominals, and let $V, V'$ be the valuations that send $p$ to $\{w_1, w_2\}$ and $\{w_1, w_2, w_3\}$ respectively. By means of a simple bisimulation argument, one can show that $\langle \mathfrak{G}, V, w_1 \rangle$ and $\langle \mathfrak{G}, V', w_1 \rangle$ are indistinguishable for $\mathcal{H}(E)$ and $\mathcal{M}_4(D)$. They are however distinguished by $\exists x y z (P x \land P y \land P z \land x \neq y \land x \neq z \land y \neq z)$. $\dashv$
Proposition 5.1 can be used to derive more proof-theoretically oriented results. Kuijper and Viana [13] present a basic axiomatization $K_{H(\emptyset)}$ for $H(\emptyset)$ which they prove to be complete for extensions with purely modal Sahlqvist axioms: for all sets $\Sigma$ of Sahlqvist formulas of the basic modal language, $K_{H(\emptyset)}\Sigma$ is complete for the class of frames defined by $\Sigma$. They also give an axiomatization $K_{H(E)}$ for $H(E)$ which is again complete for arbitrary extensions with purely modal Sahlqvist axioms.

For $M(\Delta)$ the situation is somewhat complicated, but Venema [17] presented nice results for $M(D)$. Without going into details, Venema presented a Gabbay-style axiomatization $K_{M(D)}$ that is again complete for arbitrary extensions with Sahlqvist axioms. By combining these completeness results with Proposition 5.1, we obtain the following.

**Corollary 5.1.** Let $\Sigma$ be any set of purely modal Sahlqvist axioms.

1. If $K_{H(E)}\Sigma$ is consistent, then $K_{H(E)}\Sigma$ lacks interpolation.
2. If $K_{M(D)}\Sigma$ is consistent, then $K_{M(D)}\Sigma$ lacks interpolation.

Incidentally, these results need not hold for frame classes definable using the difference operator or nominals. A simple counterexample is the class of frames with exactly one world, which is defined by the $M(D)$-formula $D > 0$ and by the $H(E)$-formula $i$. It is not hard to see that on this frame class both languages have full first-order expressivity, and hence interpolation.

**5.2. There is no decidable hybrid language with interpolation.** Areces, Blackburn, and Marx [2] pose the question whether there is any decidable modal language extending $H(\emptyset)$. Since $H(\emptyset)$ is known to be undecidable [2], Theorem 4.1.1 suggests a negative answer. Indeed, there is no decidable hybrid language with interpolation, provided we make one extra natural requirement on the language in question.

**Definition 5.1.** A modal language $(L, \models_L)$ is a concrete extension of $H(\emptyset)$ if the following conditions hold.

1. For all signatures $\sigma = (PROP, NOM)$, $L[\sigma]$ is a set of finite strings over some alphabet that includes $PROP \cup NOM \cup \{\land, \lor, \neg, \Diamond, \Box, \emptyset, (, )\}$. Furthermore, $\top \in L[\sigma]$, $PROP, NOM \subseteq L[\sigma]$, and for all $\phi, \psi \in L[\sigma]$ and $i \in NOM$ it holds that $(\phi \land \psi), \neg \phi, \Diamond \phi, \Box_i \phi \in L[\sigma]$.

2. $\models_L$ gives the usual interpretation to the proposition letters, nominals, boolean connectives, the diamond and the @-operators. I.e., $M, w \models_L \neg \phi$ iff $M, w \not\models L \phi$, etc.

The following proof is a slight generalization of the undecidability argument for $H(\emptyset)$ given by Areces, Blackburn, and Marx [2].

**Theorem 5.1.** Assume we have at least one unary modality. Every concrete extension of $H(\emptyset)$ that has interpolation on the class of all frames has an undecidable satisfiability problem on the class of all frames.

**Proof.** The proof proceeds by reduction of an undecidable problem. For any frame class $F$ and modal formula $\phi$, we say that $\phi$ is globally satisfiable on $F$ if there is a model $M$ based on a frame in $F$ such that for all world $w$ of $M$, $M, w \models \phi$. Spaan [15] proved that global satisfiability of modal formulas on
F_{23}, i.e., the class of frames in which every point has at most two successors, is undecidable. We will show that this problem can be reduced to the satisfiability problem for our hybrid language L.

**Spy-Point Lemma** (Areces, Blackburn, and Marx [2]). Let Σ be any set of first-order frame conditions and let φ be any modal formula. Then φ is globally satisfiable on the class of frames satisfying Σ iff \{Spy, □(φ⁻¹), ψ* | ψ ∈ Σ\} is satisfiable, where, for some fixed nominal i,

\[
\begin{align*}
T^{-i} & = T & (T)^* & = T \\
p^{-i} & = p & (Rx)^* & = □(x \land □y) \\
x^{-i} & = x & (x = y)^* & = □(x \land y) \\
(\neg φ)^{-i} & = \neg(φ^{-i}) & (\neg φ)^* & = \neg(φ^*) \\
(φ \land ψ)^{-i} & = φ^{-i} \land ψ^{-i} & (φ \land ψ)^* & = φ^* \land ψ^* \\
(□φ)^{-i} & = □(\negφ \land φ^{-i}) & (\exists x, φ)^* & = □(x \land φ^*)
\end{align*}
\]

\[
Spy = i \land □¬i \land □□i \land □□□y(¬i \rightarrow □(i \land □y))
\]

Clearly, the class of frames F_{23} can be defined by a first-order formula ψ_{23}. By the Spy-Point Lemma, it follows that for all modal formulas φ ∈ L[σ], φ is globally satisfiable on F_{23} iff Spy ∧ (ψ_{23})^* ∧ □(φ^-i) is satisfiable.

By Lemma 4.2, Spy ∧ (ψ_{23})^* it is equivalent to some formula γ ∈ L[σ ∪ \{i\}]. Since L is a concrete extensions of H(□), □(φ^-i) ∈ L[σ ∪ \{i\}] and L[σ ∪ \{i\}] is closed under conjunction. The translation from φ to □(φ^-i) is effective (in fact linear), and the formula γ is constant: it doesn’t depend on φ. Therefore, we have effectively reduced the global satisfiability problem for modal formulas on F_{23} to the satisfiability problem of L. It follows that satisfiability for L is undecidable.

### 5.3. Relation algebra

Relation algebra can be seen as a modal language, in which case it is often called **arrow logic**. The language is simply the basic modal language BML over a collection of three modalities: a binary modality o, a unary modality ⊗, and a nullary modality δ. Thus, the formulas of arrow logic are given by φ := p | T | ¬φ | φ ∧ ψ | φ o ψ | ⊗φ | δ. The semantics is as one would expect, in terms of frames with three accessibility relations, one for each modality. However, one focuses on a specific class of frames SQ, which consists of the frames Φ = (W, R_o, R_⊗, R_δ) where W = U × U for some set U, such that R_o = \{((w, v), (w, u), (u, v)) | w, v, u ∈ U\} (i.e., R_o denotes composition), R_⊗ = \{((w, v), (v, w)) | w, v ∈ U\} (i.e., R_⊗ denotes inverse) and R_δ = \{w, w | w ∈ U\} (i.e., R_δ denotes the identity relation on U). The class of representable relation algebras corresponds precisely to arrow logic on the frame class SQ.

Arrow logic happens not to have interpolation on SQ, and therefore Marx [14] has proposed an extension of it, RL\downarrow, which he showed to be equally expressive as first-order logic on SQ, i.e., L^1− ⊆ SQ RL\downarrow and RL\downarrow ⊆ SQ L^1−. Since SQ is itself a first-order definable frame class, it follows (as shown by Marx) that RL\downarrow has interpolation on SQ.

Using Theorem 4.1, we can show that RL\downarrow is in fact the smallest extension of arrow logic with interpolation. First of all, notice that on SQ, the difference
operator (and consequently, the global modality) is definable: $D\phi$ is equivalent to $(\neg\delta \circ \phi \circ \top) \lor (\top \circ \phi \circ \neg\delta)$. It follows that $\mathcal{M}(D) \subseteq \mathcal{BML}$ and therefore we obtain the following corollary of Theorem 4.1.3.

**Corollary 5.2.** $RL_\downarrow$ (or, equivalently, $L^1$) is the least expressive extension of arrow logic with interpolation on $SQ$.

We can rephrase this result in more algebraic terms by observing that every elementary operation on binary relations is definable in $RL_\downarrow$.

**Corollary 5.3.** Every elementary operation on binary relations is definable in $RL_\downarrow$.

**Proof.** We give a slightly informal proof. Let $\sigma$ be a signature containing proposition letters $p_1, \ldots, p_n$, and let $\phi(P_1, \ldots, P_n, x_1, \ldots, x_m)$ be any first-order defining a map from $n$ binary relations to an $m$-ary relation. We will show that there is a formula $\phi^*$ of $RL_\downarrow[\sigma]$ defining the same operation as $\phi$. This $\phi^*$ is defined inductively as follows.

\[
\begin{align*}
(Pxy)^* &= E(x \circ p \circ y) \\
(x = y)^* &= E(x \land y) \\
\top &= \top \\
(\phi \land \psi)^* &= \phi^* \land \psi^* \\
(\neg\phi)^* &= \neg(\phi^*) \\
(\exists x.\phi)^* &= \exists x.(\delta \land (\phi^*))
\end{align*}
\]

For any model $\mathcal{M}$ based on a frame in $SQ$, and for all elements $w_1, \ldots, w_m$ of $\mathcal{M}$, it holds that $\mathcal{M} \models RL_\downarrow \phi^*[x_1: (w_1, w_1), \ldots, x_m: (w_m, w_m)]$ iff $w_1, \ldots, w_m$ stand in the $\phi$ relation in $\mathcal{M}$ (this follows by induction on $\phi$).

To finish off the argument, consider any first-order formula $\phi(P_1, \ldots, P_n, x, y)$ defining an operation from binary relations to binary relations, and let $\chi \in RL_\downarrow[\sigma]$ be the formula $\exists z.\exists x.\exists y. (z \land ((x \land \delta) \circ (y \land \delta)) \land \phi^*)$. It follows that $\mathcal{M} \models (w, v) \models \chi$ iff $w$ and $v$ stand in the $\phi$ relation.

From this, we can conclude that the only way to restore interpolation for the class of representable relation algebra by expansion, is to add the entire clone of all elementary operations on binary relations. In particular, it does not suffice to add only finitely many elementary operations, or to add only Jónsson’s $Q$-operators [16].

Similar results can be obtained for cylindric algebras of finite dimension. Again, the conclusion is that first-order logic is the least expressive extension with interpolation.

**5.4. Guarded fragment.** So far, the main results were stated in terms of abstract modal languages. However, there is another perspective in terms of fragments of first-order logic. In particular, Theorem 4.1.2 can be reformulated as follows (where $\text{FREE}(\phi)$ denotes the set of variables occurring freely in the first-order formula $\phi$):

**Corollary 5.4.** Consider first-order languages with equality and constants but without function symbols of arity $\geq 2$. Let $F$ be any fragment of first-order logic satisfying the following conditions.
1. all atomic formulas (i.e., formulas of the form $R\bar{t}$ or $t_1 = t_2$) are in $F$.
2. if $\phi, \psi \in F$ then $\neg\phi \in F$ and $\phi \land \psi \in F$.
3. if $\phi$ is an atomic formula, $\psi \in F$, $\text{FREE}(\psi) \subseteq \text{FREE}(\phi)$ and $\bar{x}$ is a sequence of variables, then $\exists\bar{x}(\phi \land \psi) \in F$.
4. $F$ has interpolation: for all $\phi, \psi \in F$ with at most one free variable, if $\models \phi \rightarrow \psi$ then there is a $\theta \in F$ such that $\models \phi \rightarrow \theta$, $\models \theta \rightarrow \psi$ and all non-logical symbols occurring in $\theta$ occur both in $\phi$ and in $\psi$.

Then every first-order sentence is equivalent to an element of $F$.

Proof. Any fragment $F$ satisfying the above requirements constitutes a modal language in the following sense. For any signature $\sigma = (\text{PROP}, \text{NOM})$, let $\sigma^*$ be the first-order signature that has PROP as its unary predicates, NOM as its constants, and that has a relation $R_\Delta$ of arity $n(\Delta) + 1$ for each $\Delta \in \Omega$ (here we assume again a fixed, given set of modalities $\Omega$). Fix a first-order variables $x$, and for all signatures $\sigma$, let $L_F[\sigma]$ be the collection of first-order formulas in the signature $\sigma^*$ that are in the fragment $F$. Furthermore, let $\mathfrak{M}, w \models_{L_F} \phi(x)$ iff $\phi(x)$ holds in $\mathfrak{M}$ conceived of as a first-order structure, interpreting $x$ as $w$ and $R_\Delta$ as the accessibility relation for $\Delta$. Then $(L_F, \models_{L_F})$ is a modal language.

From the requirements given above, it follows in fact that $L_F$ extends $H(E)$ (the proof is straightforward, by induction), and that $L_F$ has interpolation on the class of all frames. Consequently, Theorem 4.1.2 applies and we can conclude that $L^1 \subseteq L_F$. In other words, every first-order formula with at most one free variable $x$ is equivalent to a formula in the fragment $F$.

Note that this is a weakening of theorem 4.1.2, since it only applies to fragments of first-order logic, thus excluding for instance second-order or infinitary languages.

Corollary 5.4 applies to the guarded fragment as defined by Grädel [11] as well as extensions of it such as the packed fragment, loosely guarded fragment and the clique-guarded fragment, provided that constants are present in the language.

Without proof, we state two generalization of this result. Firstly, Hoogland and Marx [12] show that, while interpolation fails for the Grädel-style guarded fragment, the purely relational guarded fragment (i.e., without constants) does satisfy a weak version of interpolation that is strong enough to entail the Beth property. Corollary 5.4 can be shown to apply also to this weak version of interpolation, provided that constants are allowed again.

Secondly, in the original definition of the guarded fragment by Andréka, Benthem, and Németi [1], identity statements are not allowed as guards (i.e., all quantifiers must be guarded by atomic formulas of the form $Rt_1 \ldots t_n$). Assuming that constants are allowed, the least expressive extension of this version of the guarded fragment with interpolation is precisely what Andréka, Benthem, and Németi [1] refer to as the fragment $F3$.

5.5. Characterizations of first-order logic. Formulated in slightly less formal terms, Theorems 4.1.2 and 4.1.3 express the following characterization of first-order logic:

Among all languages extending $H(E)$, or $M(D)$, first-order logic is the unique least expressive language with interpolation.
where uniqueness is taken modulo expressive equivalence. It is interesting to compare this result to more traditional characterizations of first-order logic, and in particular to Lindström style characterizations [3]. One striking difference is that our result characterizes first-order logic “from below”, i.e., as being the least expressive of a certain class of languages. Lindström’s result on the other hand characterizes first-order logic as being maximal with respect to certain properties. Another interesting aspect of our characterization is that it centers around an eminent logical notion, namely that of interpolation, unlike Lindström’s theorem, which is based on compactness and the Löwenheim-Skolem theorem, two essentially model-theoretic properties and that don’t have a clear proof-theoretic content.

\section{Discussion.} Various modal and hybrid languages are characterized in terms of interpolation. We show that very few modal languages involving nominals or the difference operator have interpolation. We presented several applications of these results. Firstly, we showed that interpolation fails for every non-trivial extension of $K_{H(E)}$ and $K_{M(D)}$ with purely modal Sahlqvist axioms. Secondly, we showed that there is no decidable hybrid language with interpolation on the class of all frames. Finally, we showed that the only way repair interpolation for relation algebra or for the guarded fragment with constants it to extend them to full first-order logic.

Some of the main proofs make essential use of tools from hybrid logic (in particular, the \#-binder). This clearly demonstrates that hybrid logic can make contributions to the wider study of modal logic and fragments of first-order logic.

The general theme of this paper can be summarized as follows.

Find expressive, decidable extensions of the basic modal language that have interpolation.

The results reported here clearly rule out hybrid languages and modal languages involving the difference operator as candidates. Incidentally, Kuijper and Viana ([13]; p.c.) have recently shown that $H(\oplus)$ has the Beth property and that it has a weak form of interpolation, namely interpolation over proposition letters (nominals are allowed to occur freely in the interpolant, similarly to modalities). Thus, the situation with hybrid logic is not quite as hopeless as might be suggested here. A nice example of a highly expressive decidable extension of the basic modal language with interpolation is the modal $\mu$-calculus [7].

We finish with an open problem. An interesting hybrid language that we did not discuss in this paper is the extension of the basic modal language with only nominals, no $\oplus$-operators. We could call this the \textit{minimal hybrid language} $H$.

\textbf{Question 6.1. What is the least expressive modal language with interpolation extending $H$?}

We have some partial results in this direction: one can show that the language in question is at least as expressive as iteration-free PDL with intersection plus nominals, graded modalities and the $\downarrow$-operator. As an upperbound, we know that the language is at most as expressive as $H(\oplus, \downarrow)$ (in fact, one can show that $H(\oplus, \downarrow)$ is the smallest extension of $H$ having strong multi-modal interpolation).
Finally, one can also show that the language in question has an undecidable satisfiability problem, cf. Theorem 5.1.

REFERENCES


