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Axiomatizing Groenendijk's Logic of Interrogation

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Jeroen Groenendijk (1999, reprinted in this book) introduced a logic, which he called the *Logic of Interrogation* (henceforth LoI), that can be used to analyze which linguistic answers are appropriate in response to a given question. Groenendijk gave only a semantic definition of his logic. For practical applications like building question-answering systems, however, we also need to understand the *proof theory* of this logic (Monz, 2003b, Section 2.4). A better understanding of the proof theory of LoI also enables us to better grasp the empirical predictions made by the model.

In this chapter, we bridge this gap, by providing a sound and complete axiomatization for LoI. Furthermore, we will show that the entailment relation of LoI is closely related to the model-theoretic notion of *definability*. Roughly speaking, the question *Who came to the party?* entails the question *Did anybody come to the party?* in LoI because the proposition that someone came to the party is definable in terms of the property of having come to the party (in the same way that the first-order sentence $\exists x.Px$ is definable in terms of the property P). This connection between question entailment and definability also shows up in the fact that an answer to a natural-language question is typically built up from instances of the question.

Organization of this chapter Section 3.1 briefly recalls the definition of the logic LoI. Section 3.2 presents our main technical result,

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namely a connection between entailment in LoI and Beth's definability theorem for first-order logic. This result is subsequently put to use in Section 3.3, where we provide a sound and complete axiomatization of LoI.

In Section 3.4, we explain how LoI can be seen not only as a logic for reasoning about linguistic questions and answers, but also with natural interpretations in mathematics, database theory, and philosophical logic. With the application of question answering in mind, we also mention a link between optimal answers and uniform interpolation, in Section 3.5.

Unifying these investigations is our view that LoI is a logic about equivalence relations between models. Different types of equivalence relations (isomorphisms, homomorphisms, bisimulations, etc.) give rise to different notions of definability and different logics of interrogation. To illustrate this, in Section 3.6 we consider a variation of LoI, in which possible worlds may have *varying domains*, and extend our results to that setting, giving again a sound and complete axiomatization.

We conclude in Section 3.7.

3.1 The Logic of Interrogation

The Logic of Interrogation was introduced by Groenendijk (1999). Its logical language, called *QL*, is an extension of first-order logic with questions. We define it as follows.[1]

Definition 3.1 (Syntax of LoI) *QL* is the set containing $!\phi$ for every sentence ϕ of first-order logic (the *assertions* or *indicatives*), and $?\psi$ for every formula ψ of first-order logic (the *questions* or *interrogatives*).

For convenience, we will assume there are no function symbols of positive arity. (Such function symbols may be replaced by relation symbols if needed.) Notice that questions can contain free variables, whereas assertions cannot. In what follows, lowercase Greek letters $\theta, \eta, \ldots$ will denote elements of *QL* (that is, questions or assertions), and uppercase Greek letters $\Sigma, \Gamma, \ldots$ will denote finite (possibly empty) sequences of elements of *QL*.

Groenendijk defines an entailment relation $\Gamma \models \theta$ between finite sequences Γ of elements of *QL* and elements θ of *QL*. An example of

[1] Our notation slightly differs from that of Groenendijk. In Groenendijk's notation, questions are of the form $?x_1 \ldots x_n.\phi$, where $x_1 \ldots x_n$ is an enumeration of the free variables of ϕ. The advantage of our notation is that it avoids having to introduce axioms such as $?\vec{x}\phi \vdash ?\vec{y}\phi$, where $\vec{x}$ is a permutation of $\vec{y}$.

a valid entailment in Groenendijk's logic is $?Px \models ?Pj$; in words, the question *Who is going to the party?* entails the question *Is John going to the party?*.

Definition 3.2 (Semantics of LoI) A *possible worlds structure* is a triple $\mathrm{A} = (W, D, I)$, where W is a set of worlds, D is a set of individuals, and I interprets the constants and relation symbols in each world. It is required that constants are interpreted *rigidly*, that is, $I_w(c) = I_v(c)$ for all $w, v \in W$. The extension of a formula ϕ at a world w, denoted by $[[\phi]]^w$, is the set of assignments g satisfying ϕ:

$$[[\phi]]^w = \{ g \in D^{\mathrm{FV}(\phi)} \mid w, g \models \phi \}.$$

Given a possible worlds structure $\mathrm{A} = (W, D, I)$, a *context* is a transitive symmetric relation $C \subseteq W^2$ (think: partitioned subset). Contexts can be *updated* with assertions or questions in the following way.

$$C[!\phi] = \{ (w, v) \in C \mid w \models \phi \text{ and } v \models \phi \}$$
$$C[?\phi] = \{ (w, v) \in C \mid [[\phi]]^w = [[\phi]]^v \}$$

Finally, entailment is defined in terms of context update, as is usual in update semantics: The entailment $\theta_1, \ldots, \theta_n \models \eta$ is valid iff, for all possible worlds structures A and contexts C, $C[\theta_1] \cdots [\theta_n][\eta] = C[\theta_1] \cdots [\theta_n]$.

Groenendijk (1999) explains the intuition behind this semantics.

Definition 3.2 is arguably more complicated than it needs to be. In particular, we will see in Section 3.2 that the use of possible worlds can be avoided: the semantics can be reformulated purely in terms of classical first-order models. Also, the fact that the left-hand side of the entailment relation is a *sequence* of formulas rather than a *set* is not essential, as the following proposition shows.

Proposition 3.1 Let Γ and Γ' enumerate the same finite set of *QL* formulas, and let θ be any *QL* formula. Then $\Gamma \models \theta$ iff $\Gamma' \models \theta$.

Proof. As context update is defined as intersection with the update potential, the commutativity and idempotence of set intersection imply that $C[\theta][\eta] = C[\eta][\theta]$ and $C[\theta][\theta] = C[\theta]$ for all C, θ, η. The result follows by the definition of entailment. □

In other words, whenever two sequences Γ and Γ' are enumerations of the same set, they entail exactly the same elements of *QL*. For this reason, we will be a bit sloppy in what follows by identifying such Γ and Γ'.

The following proposition provides many examples of entailments that are valid in LoI.

Definition 3.3 (Developments) A *development* of a set of first-order formulas Σ is a first-order formula that is built up from elements of Σ and formulas of the form $(x = y)$ using the Boolean connectives and quantifiers. In other words, the developments of Σ are given by

$$\phi ::= \chi \mid (x = y) \mid \top \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \exists x.\phi \mid \forall x.\phi,$$

where x and y are variables and $\chi \in \Sigma$.

Proposition 3.2 If ψ is a development of $\{\, \phi_1, \ldots, \phi_n \} \cup \{\, x = c \mid c \text{ is a constant} \,\}$, then $?\phi_1, \ldots, ?\phi_n \models {?\psi}$.

3.2 The Relation with Beth's Definability Theorem

In this section we establish a precise connection between LoI and Beth's Definability Theorem for first-order logic. This connection will be used later to prove our axiomatization of LoI complete.

Definition 3.4 (Isomorphisms) Given a set Γ of first-order formulas, a Γ-*isomorphism* between two first-order models M and N is a bijection $f : \mathrm{M} \to \mathrm{N}$ between the domains of M and N such that for each formula $\phi(x_1, \ldots, x_n) \in \Gamma$, and for any sequence of individuals $d_1, \ldots d_n$ in the domain of M, we have $\mathrm{M} \models \phi[d_1, \ldots, d_n]$ iff $\mathrm{N} \models \phi[fd_1, \ldots, fd_n]$.

Definition 3.5 (Implicit Definitions) Let Σ be a first-order theory, let Γ be a set of first-order formulas, and let ψ be a first-order formula. The theory Σ *implicitly defines* the formula ψ in terms of the formulas in Γ iff every Γ-isomorphism between two models of Σ is a $\{\psi\}$-isomorphism as well.

Intuitively, a theory implicitly defines ψ in terms of Γ if the denotation of ψ is completely determined by that of Γ—in other words, if, whenever two models completely agree (are isomorphic) with respect to Γ, they completely agree (are isomorphic) with respect to ψ.

If a formula ψ is a development of some set of formulas Γ, then a simple induction over ψ shows that every theory (including the empty theory) implicitly defines ψ in terms of Γ. In a sense, Beth's Definability

Theorem tells us that the converse holds, modulo logical equivalence. (In what follows, we will write $\models_{\text{fol}}$ for classical first-order entailment, in order to distinguish it from entailment in LoI.)

Theorem 3.1 (Beth Definability) Σ implicitly defines ψ in terms of Γ iff there is a development χ of Γ, with the same free variables $\vec{x}$ as ψ, so that $\Sigma \models_{\text{fol}} \forall\vec{x}(\psi \leftrightarrow \chi)$.

We can think of this theorem as a syntactic characterization of implicit definability in first-order logic.

Some comments should be made concerning this formulation of Beth's Definability Theorem. First, this formulation is slightly stronger than Beth's original version. In the above shape, the result is sometimes referred to as the *Projective Beth Theorem*, and it was first proved by William Craig (1957). Second, in the more usual expositions, Beth's Definability Theorem is formulated in terms of atomic relation symbols and constants rather than formulas with free variables (in other words, ψ and the elements of Γ would be predicate symbols or constants rather than formulas). Our apparent generalization of the theorem can easily be obtained from this more classical version by introducing for each $\gamma(x_1, \ldots, x_n) \in \Gamma$ a new n-ary predicate P_γ and by extending Σ with formulas of the form $\forall\vec{x}(P\vec{x} \leftrightarrow \gamma(\vec{x}))$.

Here comes the connection between LoI and the Beth Definability Theorem.

Theorem 3.2 The LoI entailment $!\phi_1, \ldots, !\phi_n, ?\chi_1, \ldots, ?\chi_m \models ?\psi$ holds iff the set of asserted formulas $\{\phi_1, \ldots, \phi_n\}$ implicitly defines ψ in terms of $\Gamma = \{\chi_1, \ldots, \chi_m\} \cup \{\, x = c \mid c \text{ is a constant}\,\}$.

Proof. [⇒] Suppose $!\phi_1, \ldots, !\phi_n, ?\chi_1, \ldots, ?\chi_m \models ?\psi$. Let $f : \mathrm{M} \rightarrow \mathrm{N}$ be a Γ-isomorphism between $\mathrm{M} = (D, I)$ and $\mathrm{N} = (D', I')$, both models of $\{\phi_1, \ldots, \phi_n\}$. We will show that f is a $\{\psi\}$-isomorphism as well. Define the possible worlds structure $\mathrm{A} = (\{w, v\}, D, I'')$, where $I''_w = I$ and $I''_v = f^{-1} \circ I' \circ f$. (The construction of Γ guarantees that all constants have a rigid interpretation in A.) Furthermore, let C be the ("trivial") context $\{(w, w), (w, v), (v, w), (v, v)\}$. A simple inductive argument shows that $C[!\phi_1] \cdots [!\phi_n][?\chi_1] \cdots [?\chi_m] = C$. From the LoI entailment assumed at the start, it follows that $C[?\psi] = C$, from which we can conclude that $[[\psi]]^w = [[\psi]]^v$. Therefore, f must be a $\{\psi\}$-isomorphism.

[$\Leftarrow$] Suppose $\{\phi_1, \ldots, \phi_n\}$ implicitly defines ψ in terms of Γ, and consider any possible worlds structure $\mathrm{A} = (W, D, I)$ and context C. Let C' be the updated context $C[!\phi_1]\cdots[!\phi_n][?\chi_1]\cdots[?\chi_m]$. We will show that $C'[?\psi] = C'$. Consider any $(w, v) \in C'$. A simple inductive argument shows that $w \models \phi_i$ and $v \models \phi_i$ for all $i \leq n$. Similarly, $[[\chi_i]]^w = [[\chi_i]]^v$ for all $i \leq m$. It follows that the identity relation on D is a Γ-isomorphism between A_w and A_v. By the definition of implicit definition, the identity relation on D is a $\{\psi\}$-isomorphism between A_w and A_v. In other words, $[[\psi]]^w = [[\psi]]^v$. Therefore, $(w, v) \in C'[?\psi]$. $\square$

By combining Theorem 3.1 and Theorem 3.2, we obtain a syntactic characterization of the LoI entailment relation when the consequent is a question.

Corollary 3.1 The LoI entailment $!\phi_1, \ldots, !\phi_n, ?\chi_1, \ldots, ?\chi_m \models ?\psi$ holds iff there is a development ψ' of $\{\chi_1, \ldots, \chi_m\} \cup \{\, x = c \mid c \text{ is a constant}\,\}$, with the same free variables $\vec{x}$ as ψ, so that $\phi_1, \ldots, \phi_n \models_{\mathrm{fol}} \forall\vec{x}(\psi \leftrightarrow \psi')$.

Corollary 3.1 characterizes LoI validity in the case where the consequent is a question. As it turns out, the validity of entailments with an assertion as the consequent can be characterized in even simpler terms.

Theorem 3.3 $!\phi_1, \ldots, !\phi_n, ?\chi_1, \ldots, ?\chi_m \models !\psi$ iff $\phi_1, \ldots, \phi_n \models_{\mathrm{fol}} \psi$.

Proof. [$\Rightarrow$] Suppose $!\phi_1, \ldots, !\phi_n, ?\chi_1, \ldots, ?\chi_m \models !\psi$, and let $\mathrm{M} = (D, I)$ be a first-order model that verifies ϕ_i for all $i \leq n$. We will show that M verifies ψ as well. Construct the possible worlds structure $\mathrm{A} = (\{w\}, D, I')$, where $I'_w = I$. Furthermore, let C be the context $\{(w, w)\}$. A simple inductive argument shows that $C[!\phi_1]\cdots[!\phi_n][?\chi_1]\cdots[?\chi_m] = C$. From the LoI entailment assumed at the start, it follows that $C[!\psi] = C$, from which we can conclude that $w \models \psi$. Therefore, $\mathrm{M} \models \psi$.

[$\Leftarrow$] Suppose $\phi_1, \ldots, \phi_n \models_{\mathrm{fol}} \psi$, and let $\mathrm{A} = (W, D, I)$ be any possible worlds structure and C any context. Let C' be the updated context $C[!\phi_1]\cdots[!\phi_n][?\chi_1]\cdots[?\chi_m]$. We will show that $C'[!\psi] = C'$. Consider any $(w, v) \in C'$. A simple inductive argument shows that $w \models \phi_i$ and $v \models \phi_i$ for all $i \leq n$. From the first-order entailment assumed at the start, it follows that $w \models \psi$ and $v \models \psi$. Therefore, $(w, v) \in C'[!\psi]$. $\square$

In other words, if the conclusion is an assertion, then validity in LoI reduces to classical first-order validity. This means that LoI is a conservative extension of classical first-order logic: if one restricts attention to assertions, validity in LoI and classical validity coincide.

Theorems 3.2 and 3.3 give us an alternative semantics for LoI that makes no reference to possible worlds. Intuitively, the reason that such a semantics exists is that, to test an LoI entailment, it suffices to consider structures with only two possible worlds.

3.3 Axiomatization

Table 1 lists a sound and complete axiomatization of LoI. When axiomatizing *classical* first-order logic, it suffices to axiomatize the first-order *tautologies*: by *compactness* and the *deduction theorem*, an entailment $\Sigma \models_{\text{fol}} \phi$ holds just in case there are $\psi_1, \ldots, \psi_n \in \Sigma$ such that $\models_{\text{fol}} \psi_1 \wedge \cdots \wedge \psi_n \rightarrow \phi$. In LoI, on the other hand, there is no easy way to reduce the entailment problem to the problem of validity of formulas, due to the absence of a suitable analogue of the deduction theorem. For this reason, Table 1 axiomatizes the entailment relation rather than just the tautologies.[2]

The axiom scheme [CT] expresses that LoI is an extension of first-order logic; in other words, every valid first-order sentence is still valid in LoI. In fact, we know already from Theorem 3.3 that LoI is a conservative extension of first-order logic. Interestingly, there is a second way in which LoI is conservative over first-order logic, namely with regards to *structural rules*. Johan van Benthem (1996, Chapter 7) has shown that the [Ref], [Cut], [Monotonicity], [Permutation] and [Contraction] completely characterize the structural properties of classical entailment. Table 1 shows that LoI is conservative over first-order logic, in the sense that these structural properties are still valid.

Theorem 3.4 The axiomatization in Table 1 is sound and complete for entailment in LoI. That is, for all Γ and θ, $\Gamma \vdash \theta$ iff $\Gamma \models \theta$.

Proof. Soundness is straightforward. As for completeness, suppose $\Gamma \models \theta$, where Γ is some sequence $!\phi_1, \ldots, !\phi_n, ?\chi_1, \ldots, ?\chi_m$. We can distinguish two cases.

[2] One could consider introducing an explicit implication sign $\Longrightarrow$ that operates on questions and assertions. With the help of such a connective, $!\phi \models ?\psi$ could for instance be reduced to $\models (!\phi \Longrightarrow ?\psi)$. The proper semantics of this connective is not clear, but it might be worth investigating the intuitive connection with *conditional questions*, such as *If we go for dinner tonight, will Mary join us?* (Velissaratou, 2000).

TABLE 1 Axiomatization of LoI

	[CT]	$!\phi_1, \ldots, !\phi_n \vdash !\psi$	whenever $\phi_1, \ldots, \phi_n \models_{\text{fol}} \psi$
+	[$\top$]	$\vdash ?\top$	
	[$\neg$]	$?\phi \vdash ?\neg\phi$	
	[$\wedge$]	$?\phi, ?\psi \vdash ?(\phi \wedge \psi)$	
+	[$\vee$]	$?\phi, ?\psi \vdash ?(\phi \vee \psi)$	
	[$\exists$]	$?\phi \vdash ?\exists x \phi$	
+	[$\forall$]	$?\phi \vdash ?\forall x \phi$	
+	[Subst]	$?\phi \vdash ?\phi[x/t]$	where t is substitutable for x in ϕ
	[Equality]	$\vdash ?(x = y)$	
	[Const]	$\vdash ?(x = c)$	
	[Equiv]	$!\forall \vec{x}(\phi \leftrightarrow \psi), ?\phi \vdash ?\psi$	where $\text{FV}(\phi) \cup \text{FV}(\psi) = \{\vec{x}\}$
+	[Ref]	$\theta \vdash \theta$	
	[Cut]	If $\Gamma \vdash \theta$ and $\Gamma, \theta \vdash \eta$ then $\Gamma \vdash \eta$	
	[Monotonicity]	If $\Gamma \vdash \theta$ then $\Gamma, \Gamma' \vdash \theta$	
	[Permutation]	If $\Gamma, \theta, \eta, \Gamma' \vdash \zeta$ then $\Gamma, \eta, \theta, \Gamma' \vdash \zeta$	
+	[Contraction]	If $\Gamma, \theta, \theta \vdash \eta$ then $\Gamma, \theta \vdash \eta$	

Axioms marked with + are derivable

- θ is of the form $!\psi$. By Theorem 3.3, $\phi_1, \ldots, \phi_n \models_{\text{fol}} \psi$. By the axiom [CT], $!\phi_1, \ldots, !\phi_n \vdash !\psi$. By the structural rule [Weakening], it follows that $\Gamma \vdash !\psi$.
- θ is of the form $?\psi$. By Corollary 3.1, there is a formula ψ' such that (1) $\phi_1, \ldots, \phi_n \models_{\text{fol}} \forall\vec{x}(\psi \leftrightarrow \psi')$ and (2) ψ' is a development of $\{\chi_1, \ldots, \chi_n\} \cup \{\, x = c \mid c \text{ is a constant}\,\}$. From (1), the axiom [CT] and the structural rule [Weakening], it follows that $\Gamma \vdash\, !\forall\vec{x}(\phi \leftrightarrow \phi')$. From (2), it follows that $\Gamma \vdash ?\phi'$ (by induction on ϕ'). Therefore, $\Gamma \vdash ?\phi$.

□

3.4 Four Perspectives on LoI

LoI was introduced as a logic for reasoning about the relevance of linguistic utterances. However, besides this linguistic view on the logic, there are alternative perspectives. LoI has a natural mathematical interpretation (as a logic describing equivalence relations between models), a computational interpretation (describing reductions among database queries), and a philosophical interpretation (describing logicality of operations). We survey the different perspectives on LoI in this section.

LoI provides not only a unifying perspective on these different topics, but also a common starting point for exploring variations on them: different notions of utterance relevance, model equivalence, query reduction, and logicality correspond to different variants of LoI. These in turn give rise to variations of Beth's Definability Theorem. Section 3.6 presents one such variation: a variant of LoI defined in terms of possible worlds with varying domains.

3.4.1 Linguistic

Groenendijk (1999) introduced LoI as a logic for reasoning about relevance (*aboutness*) of linguistic utterances. For instance, that *John is going to the party* is a relevant response to the question *Who is going to the party?* is reflected by the validity of $?Px \models ?Pj$. In general, it is claimed, an assertion $!\phi$ is relevant after a sequence of utterances Γ (Γ *licenses* $!\phi$, to use Groenendijk's terminology) if and only if $\Gamma \models ?\phi$. This claim has several empirical problems, and consequently, slight variations of the semantics have been proposed that arguably make better predictions.

For instance, Groenendijk and Stokhof (1997) discuss an alternative semantics for LoI in terms of possible worlds structures with *varying domains*. This more liberal semantics allows each possible world to have a different domain. Consequently, the question $?Px$ no longer entails $?\neg Px$, thus invalidating the axiom [$\neg$] in Table 1. We will study this alternative semantics in more detail, and give a sound and complete axiomatization, in Section 3.6.

Another parameter of variation concerns the interpretation of terms. Following Groenendijk (1999), we treat constants as rigid designators. Consequently, the question $?Px$ entails the question $?Pj$; it is reflected in Table 1 as the axiom [Const]. The logic would be quite different if some or all constants were non-rigid. An even more fine-grained semantics, introduced by Aloni (2001), interprets constants as elements of *conceptual covers*. Again, this change of semantics crucially affects the validities.

One important feature of sound and complete axiomatizations like the one provided in Section 3.3 is that they provide further insight in the precise implications of the different "design choices".

3.4.2 Mathematical

From a more mathematical point of view, LoI can be seen as a logic of isomorphisms. For instance, the entailment $?Px, ?Qx \models ?(Px \wedge Qx)$ (which can be proved using Proposition 3.2 or Theorem 3.2) roughly says that every $\{Px, Qx\}$-isomorphism is a $\{Px \wedge Qx\}$-isomorphism. Likewise, the entailment $?Px, !\forall x(Px \leftrightarrow Qx) \models ?Qx$ means that every $\{Px\}$-isomorphism between models satisfying $\forall x(Px \leftrightarrow Qx)$ is a $\{Qx\}$-isomorphism. Continuing on this observation, we can see LoI as a logic for describing isomorphisms.

If isomorphisms play such a central part in this logic, what happens if we replace them by *homomorphisms* or *bisimulations*? We will come back to this question.

3.4.3 Computational

A third perspective on the LoI is as a logic for reasoning about database queries. In database research, there is much interest in reasoning about query equivalence. For example, the two queries $?Px$ and $?(Px \wedge Qx) \vee (Px \wedge \neg Qx)$ give identical outcomes, but the former is easier to process than the latter. A less trivial example is the following: if we know that $\forall x(Px \to Qx)$ holds in our database, then the queries $?(Px \wedge Qx)$ and $?Px$ are equivalent. This is reflected in the two LoI entailments

$$!\forall x(Px \to Qx), ?Px \models ?(Px \wedge Qx),$$
$$!\forall x(Px \to Qx), ?(Px \wedge Qx) \models ?Px.$$

Thus, LoI is a logic of database queries—albeit one that builds on a rather crude notion of query equivalence, since it considers $?Px$ and $?\neg Px$ equivalent. In fact, Theorem 3.2 suggests that we should interpret $!\phi_1, \ldots, !\phi_n, ?\chi_1, \ldots, ?\chi_m \models ?\psi$ roughly as *The answer to* $?\psi$ *can be computed from the answers to* $?\chi_1, \ldots, ?\chi_m$*, given that* $\phi_1, \ldots, \phi_n$ *are true*. This computational intuition behind LoI can be made more explicit.

Proposition 3.3 Suppose $!\phi_1, \ldots, !\phi_n, ?\chi_1, \ldots, ?\chi_m \models ?\psi$. Then the extension of ψ in a finite model satisfying $\phi_1, \ldots, \phi_n$ can be effectively computed from the extension of $\chi_1, \ldots, \chi_m$, the domain of the model, and the interpretation of all constants. In fact, the computation can be performed in PSPACE.

Proof. By Corollary 3.1, there is a development ψ' of $\chi_1, \ldots, \chi_n$ such that

$$\phi_1, \ldots, \phi_n \models_{\text{fol}} \forall \vec{x}(\psi \leftrightarrow \psi').$$

Since ψ' is built up from instances of $\chi_1, \ldots, \chi_n$, we can apply any PSPACE model checking algorithm for first-order logic to compute the extension of ψ' (hence of ψ) from the extension of $\chi_1, \ldots, \chi_n$, the domain, and the interpretation of the constants. □

The converse does not hold, due to the restriction to finite models. For instance, let ψ be a first-order sentence that states that some binary relation $<$ is an unbounded strict order. Because ψ only has infinite models, there is trivially an efficient algorithm that computes the truth value of ψ in any given finite model—just report *False* right away. Nevertheless, $\not\models ?\psi$, since there are models of equal (infinite) cardinality such that ψ is true in one and false in the other. Thus, one could say that LoI is sound but not complete for this "computational" or "finite-domain" semantics.

3.4.4 Philosophical

Tarski (1986) asks the important philosophical question: *What counts as a logical operation?* Our intuitions say that predicate intersection is a logical operation, whereas intersection-if-it-rains-and-union-otherwise is not a logical operation. The question, then, is: *what relevant feature of the former operation does the latter lack?* Tarski gives an answer in terms of bijections: predicate intersection is logical because for any bijection f between the domains of two models, and for any two predicates P, Q, if f respects membership of P and Q, then it also respects membership of their intersection. The operation intersection-if-it-rains-and-union-otherwise does not satisfy this property, and hence is not logical according to Tarski. Tarski's solution is not undisputed. For instance, Feferman (1999) argues that in order for an operation to be truly logical, the above criterion should hold not only for bijections but for any surjective function.

Tarski's criterion is closely connected to the notion of implicit definitions, as given in Definition 3.5. To make this connection precise, consider any first-order formula ϕ with free variables $x_1, \ldots, x_n$ and containing predicate symbols $P_1, \ldots, P_m$ (with $n, m \geq 0$). For convenience, assume that ϕ contains no constants. Each such formula defines an operation on relations, namely

$$\lambda P_1 \ldots P_m. (\lambda x_1 \ldots x_n. \phi).$$

This operation takes as input m relations of appropriate arity, and outputs a single n-ary relation. Now, it follows from Definition 3.5 and Theorem 3.2 that this operation is logical in Tarski's sense iff the LoI entailment

$$?P_1\vec{x}_1, \ldots, ?P_n\vec{x}_n \models ?\phi$$

holds (where each $\vec{x}_i$ is a sequence of as many variables as the arity of P_i).

On the one hand, this connection gives us a further motivation for the current definition of LoI: it seems plausible that all and only the *logical* operations may be used to compose (develop) complex questions and answers out of simple ones. On the other hand, variants of LoI may be obtained by considering alternative notions of logicality such as the one proposed by Feferman. These variants of LoI will have different predictions as to what constitutes an appropriate answer to a question.

3.5 Interlude: Optimal Answers and Uniform Interpolation

Recall from Section 3.4.1 that an assertion $!\psi$ is a relevant response to (*is licensed by*) a question $?\phi$ in case $?\phi \models ?\psi$. Suppose we are given a question $?\phi$ and a certain amount of knowledge, represented by a finite set of assertions Σ. One would like to find an assertion $!\psi$, called an *optimal answer to* $?\phi$ *given* Σ, satisfying

1. $\Sigma \models !\psi$. ("$!\psi$ follows from Σ")
2. $?\phi \models ?\psi$. ("$!\psi$ is a relevant response to $?\phi$")
3. For all assertions $!\psi'$ satisfying 1 and 2, $!\psi \models !\psi'$. ("$!\psi$ is most informative")

Is there always an optimal answer to $?\phi$ given the information in Σ? The answer is *No.* Let Σ be any satisfiable theory (that is, any set of assertions) that has only infinite models, and set the question $?\phi$ to simply $?\top$. Let ψ_n (for each $n \in \mathbb{N}$) be a sentence that says that there are at least n different objects. Then the entailments $\Sigma \models \psi_n$ and $?\phi \models ?\psi_n$ hold for all $n \in \mathbb{N}$. (The latter entailment holds because the domain of individuals is fixed across possible worlds in the semantics of LoI.) In other words, the potential answers ψ_n all satisfy conditions 1 and 2 above. Any optimal answer ψ, then, must entail each ψ_n and furthermore be equivalent to a formula that contains no non-logical symbols. It follows (e.g., using the invariance of first-order formulas for potential isomorphisms) that $\neg\psi$ defines the class of finite models, which is known not to be first-order definable.

The reader might have noticed that this proof is almost identical to Henkin's proof that first-order logic does not have *uniform interpolation* (Henkin 1963). Indeed, these two properties—uniform interpolation and there being always a unique most informative answer—are intimately related. Making the connection precise may require a fair amount of abstract model theory, so we avoid it here.

3.6 A Variation on LoI: Varying Domains

We now turn to a variation of LoI that was more or less proposed by Groenendijk and Stokhof (1997). The only difference from LoI as defined in Section 3.1 is that we drop the restriction to possible worlds structures with *constant domains*. We will show that this variant of LoI admits an analysis analogous to the one already given.

Definition 3.6 (LoI semantics, varying domains) A *varying domain structure* is a quadruple $\mathrm{A} = (W, Dom, D, I)$, where W is a set of worlds, Dom is a set of entities, $D : W \to \wp(Dom)$ assigns a domain to each world, and I interprets the constants and relations in each world, such that

- For each constant c and world w, $I_w(c) \in D_w$.
- For each n-ary relation R and world w, $I_w(R) \subseteq D_w^n$.
- All constants are rigid: for all constants c and worlds w, v, $I_w(c) = I_v(c)$.

(This implies that the intersection of the domains is non-empty.) The extension of a formula ϕ at a world w, denoted by $[[\phi]]^w$, is

$$[[\phi]]^w = \{\, g \in D_w^{\mathrm{FV}(\phi)} \mid w, g \models \phi \,\}.$$

Contexts, updates and entailment are defined as before, with the single difference that varying domain structures replace constant domain structures.

We use the notation $\models_{\mathrm{vd}}$ for LoI entailment under the varying domain semantics. Note that every constant domain structure is a varying domain structure, so $\Gamma \models \theta$ whenever $\Gamma \models_{\mathrm{vd}} \theta$. In order to axiomatize this variant of LoI, we need to find an appropriate analogue of Beth's Definability Theorem. In order to do so, we need to introduce a new kind of model equivalence relation.

Definition 3.7 (Full Partial Isomorphisms) Given a set of first-order formulas Γ, a *partial* Γ*-isomorphism* between two first-order models M and N is an injective partial function $f : \mathrm{M} \to \mathrm{N}$ such that for each formula $\phi(x_1, \ldots, x_n) \in \Gamma$, where $x_1, \ldots, x_n$ are the free variables of ϕ, and for any individuals $d_1, \ldots, d_n$ in the domain of f, we have $\mathrm{M} \models \phi[d_1, \ldots, d_n]$ iff $\mathrm{N} \models \phi[fd_1, \ldots, fd_n]$. In addition, f is a *full* partial Γ-isomorphism (FPI) if the following conditions hold:

- For any formula $\phi(x_1, \ldots, x_n)$ in Γ and individuals $d_1, \ldots, d_n$ in the domain of M, if $\mathrm{M} \models \phi[d_1, \ldots, d_n]$ then $d_1, \ldots, d_n$ are in the domain of f.
- For any formula $\phi(x_1, \ldots, x_n)$ in Γ and individuals $d'_1, \ldots, d'_n$ in the domain of N, if $\mathrm{N} \models \phi[d'_1, \ldots, d'_n]$ then $d'_1, \ldots, d'_n$ are in the range of f.

Definition 3.8 (Implicit Definitions over FPIs) Let Σ be a first-order theory, let Γ be a set of first-order formulas, and let ψ be a first-order formula. The theory Σ *implicitly defines* the formula ψ in terms of the formulas in Γ over full partial isomorphisms iff every full partial Γ-isomorphism between two models of Σ is a full partial $\{\psi\}$-isomorphism as well.

Definition 3.9 (Strict Developments) The *strict development* of a set of first-order formulas Γ are the first-order formulas given by

$$\phi ::= \chi \mid (x = y) \wedge \psi \mid \top \mid \neg\phi_1 \wedge \phi_2 \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \exists y.\phi,$$

where $\chi \in \Gamma$, $x \in \mathrm{FV}(\psi)$ in the case of equality, $\mathrm{FV}(\phi_1) \subseteq \mathrm{FV}(\phi_2)$ in the case of negation, and $\mathrm{FV}(\phi_1) = \mathrm{FV}(\phi_2)$ in the case of disjunction.

If a formula ψ is a strict development of some set of formulas Γ, then a simple induction over ψ shows that the trivial theory implicitly defines ψ in terms of Γ over full partial isomorphisms. In a sense, the following variation on Beth's Definability Theorem tells us that the converse holds, modulo logical equivalence as before.

Theorem 3.5 (Beth Definability for FPIs) Σ implicitly defines ψ in terms of Γ over full partial isomorphisms iff there is a strict development χ of Γ, with the same free variables $\vec{x}$ as ψ, so that $\Sigma \models_{\text{fol}} \forall\vec{x}(\psi \leftrightarrow \chi)$.

The Beth definability theorem is usually proved as a corollary of the Craig interpolation theorem. For the proof of Theorem 3.5, we will need the following more refined version of interpolation Otto (2000):

Lemma 3.1 (Relativized interpolation) Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a collection of unary predicates, and let ϕ, ψ be $\mathcal{U}$-relativized formulas (i.e., formulas in which all quantification is of the form $\exists x.(Ux \wedge \cdots)$ or $\forall x.(Ux \to \cdots)$, with $U \in \mathcal{U}$). Furthermore, suppose that $\models \phi \to \psi$. Then there is a $\mathcal{U}$-relativized formula ξ such that:

1. $\models \phi \to \xi$ and $\models \xi \to \psi$.
2. All free variables of ξ are free variables of ϕ and of ψ.
3. All relation symbols occurring in ξ occur both in ϕ and in ψ.

Proof of Theorem 3.5. We prove only the hard direction ($\Rightarrow$). We discuss only the simplified case where Γ consists of atomic relation symbols: the general result can be derived by introducing a new atomic relation symbol R_χ for each $\chi \in \Gamma$ and extending Σ with $\forall\vec{x}(R_\chi\vec{x} \leftrightarrow \chi(\vec{x}))$. Let

$$\theta_\Gamma(y) = \bigvee_{\substack{R\in\Gamma \\ k\leq \text{arity}(R)}} \exists x_1 \ldots x_{\text{arity}(R)}.(R(\vec{x}) \wedge x_k = y)$$

which expresses that y belongs to a tuple in the denotation of some $R \in \Gamma$. Two straightforward inductive arguments, one in each direction, verify the following fact:

> *Fact 1.* Up to logical equivalence, the strict developments of Γ are precisely the first-order formulas of the form $\zeta(x_1, \ldots, x_n) \wedge \theta_\Gamma(x_1) \wedge \cdots \wedge \theta_\Gamma(x_n)$, where ζ is built up from atomic relation symbols in Γ using the Boolean connectives and θ_Γ-relativized quantification.

Now, suppose Σ implicitly defines ψ in terms of $\Gamma = \{R_1, \ldots, R_n\}$ over FPIs. For a start, it follows quite easily from the definition of implicit definability for FPIs (in particular, from the requirements on the domain and range of FPIs) that

> *Fact 2.* $\Sigma \models \forall\vec{x}(\psi(\vec{x}) \to \bigwedge_k \theta_\Gamma(x_k))$

Next, let Σ' be a copy of Σ in which each relation symbol $R \notin \Gamma$ is uniformly replaced by a new relation symbol R' of the same arity. Furthermore, pick new unary predicates P_1, P_2, U. Let Σ^{P_1} and Σ'^{P_2} be obtained from Σ and Σ' by relativizing all quantifiers by P_1 or P_2, respectively. Consider the theory

$$\Sigma^* = \Sigma^{P_1} \cup \Sigma'^{P_2} \cup \{\forall x.(Ux \to P_1 x)\} \cup \{\forall x.(Ux \to P_2 x)\} \\ \cup \{\forall \vec{x}((\bigwedge_k P_i x_k) \wedge R\vec{x} \to \bigwedge_k Ux_k) \mid R \in \Gamma; i = 1, 2\}$$

This theory Σ^* describes a situation where there are two models of Σ and a full partial Γ-isomorphism between them: P_1 and P_2 define the domains of the two models, and the full partial isomorphism is the identity function on U. That the FPI need not preserve relations $R \notin \Gamma$ is reflected in the fact that the submodel defined by P_1 satisfies Σ whereas the submodel defined by P_2 satisfies Σ'.

Since Σ implicitly defines ψ in terms of Γ, it follows that

$$\Sigma^* \models (\psi^{P_1}(\vec{x}) \wedge \bigwedge_k P_1 x_k) \to (\psi^{P_2}(\vec{x}) \wedge \bigwedge_k P_2 x_k),$$

where ϕ^{P_i} is the result of relativizing all quantifiers in ϕ by P_i. By compactness, this holds already for a finite subset of Σ. Hence, in what follows, we will assume that Σ (and hence also Σ') is finite. By writing Σ^{P_1} and Σ'^{P_2} as large conjunctions and rearranging the formulas in the above entailment, we obtain

$$\models (\bigwedge \Sigma^{P_1}) \wedge \forall x.(Ux \to P_1 x) \wedge \forall \vec{x}((\bigwedge_k P_1 x_k) \to \bigwedge_{R\in\Gamma}(R\vec{x} \to \bigwedge_k Ux_k)) \wedge \\ (\psi^{P_1}(\vec{x}) \wedge \bigwedge_k P_1 x_k) \\ \to \\ \Big(((\bigwedge \Sigma'^{P_2}) \wedge \forall x.(Ux \to P_2 x) \wedge \forall \vec{x}((\bigwedge_k P_2 x_k) \to \bigwedge_{R\in\Gamma}(R\vec{x} \to \bigwedge_k Ux_k))) \\ \to \quad (\psi^{P_2}(\vec{x}) \wedge \bigwedge_k P_2 x_k)\Big)$$

We now apply the relativized interpolation theorem (Lemma 3.1) to obtain a $\{P_1, P_2, U\}$-relativized interpolant ξ for this implication. Interpolation guarantees that only U and the relation symbols in Γ can occur in ξ. Therefore, neither P_1 nor P_2 occurs in ξ, so all quantifiers in ξ are relativized by U. As a final step, we replace all occurrences of U in ξ by θ_Γ. The resulting formula ξ' satisfies the following property:

Fact 3. $\Sigma \models \forall \vec{x}(\psi(\vec{x}) \leftrightarrow \xi'(\vec{x}))$

(This follows from the fact that ξ is entailed by the interpolation antecedent above, and entails the interpolation consequent above, by replacing P_1, P_2, and U by $\top$, $\top$, and θ_Γ, respectively).

Finally, $\xi'(\vec{x}) \wedge \bigwedge_k \theta_\Gamma(x_k)$ is the strict development of Γ that we are looking for: By Facts 2 and 3 together, $\Sigma \models \forall\vec{x}(\psi(\vec{x}) \leftrightarrow (\xi'(\vec{x}) \wedge \bigwedge_k \theta_\Gamma(x_k)))$. By Fact 1, $\xi'(\vec{x}) \wedge \bigwedge_k \theta_\Gamma(x_k)$ is (logically equivalent to) a strict development of Γ. □

Proposition 3.4 An alternative semantics for LoI with varying domains that does not use possible worlds structures is as follows.

1. $!\phi_1, \ldots, !\phi_n, ?\chi_1, \ldots, ?\chi_m \models_{\text{vd}} !\psi$ iff $\phi_1, \ldots, \phi_n \models_{\text{fol}} \psi$.
2. $!\phi_1, \ldots, !\phi_n, ?\chi_1, \ldots, ?\chi_m \models_{\text{vd}} ?\psi$ iff $\{\phi_1, \ldots, \phi_n\}$ implicitly defines ψ in terms of $\{\chi_1, \ldots, \chi_m\} \cup \{ x = c \mid c \text{ is a constant} \}$ over full partial isomorphisms.

Proof. Analogous to Theorems 3.2 and 3.3. □

Corollary 3.2 $!\phi_1, \ldots, !\phi_n, ?\chi_1, \ldots, ?\chi_m \models_{\text{vd}} ?\psi$ iff there is a strict development ψ' of $\{\chi_1, \ldots, \chi_m\} \cup \{ x = c \mid c \text{ is a constant} \}$ with the same free variables $\vec{x}$ as ψ such that $\phi_1, \ldots, \phi_n \models_{\text{fol}} \forall\vec{x}(\psi \leftrightarrow \psi')$.

The difference between developments and strict developments is reflected in different LoI entailments with constant and varying domains. For instance, the LoI entailment $?Px \models ?\forall x.Px$ (the axiom [$\forall$] in Table 1) no longer holds under the varying domain semantics. The counterexample is illustrative: let M be a first-order model with domain $\{a\}$ in which the unary predicate P has the extension $\{a\}$. Let N be a model with domain $\{a, b\}$ in which P also has the extension $\{a\}$. Furthermore, suppose all constants denote a in both M and N. Then P has the same extension in M and N, but $\forall x.Px$ does not: it is true in M and false in N.

Table 2 contains a sound and complete axiomatization of $\models_{\text{vd}}$. Completeness follows in the same way as before.

3.6.1 Four Perspectives on LoI Revisited

We saw earlier that there are four natural perspectives on LoI, namely linguistic (in terms of aboutness), mathematical (in terms of isomorphisms), computational (in terms of database queries), and philosophical (in terms of logical operations). We now consider the varying domain version of LoI from the same four perspectives.

Linguistic Table 2 shows that the varying domain version of LoI makes counter-intuitive linguistic predictions. Intuitively, *Everybody is going to the party* is a very natural answer to *Who is going to the party?*, and *Somebody is going to the party* is less natural an answer.

TABLE 2 Axiomatization of the varying domain version of LoI

	[CT]	$!\phi_1, \ldots, !\phi_n \vdash !\psi$	whenever $\phi_1, \ldots, \phi_n \models_{\text{fol}} \psi$
+	[⊤]	$\vdash ?\top$	
	[¬′]	$?\phi, ?\psi \vdash ?(\phi \wedge \neg\psi)$	where $\text{FV}(\psi) \subseteq \text{FV}(\phi)$
	[∧]	$?\phi, ?\psi \vdash ?(\phi \wedge \psi)$	
	[∨]	$?\phi, ?\psi \vdash ?(\phi \vee \psi)$	where $\text{FV}(\phi) = \text{FV}(\psi)$
	[∃]	$?\phi \vdash ?\exists x\phi$	
+	[∀′]	$?\phi, ?\psi \vdash ?\forall\vec{x}(\phi \rightarrow \psi)$	where $\text{FV}(\phi) = \{\vec{x}\}$
+	[Subst]	$?\phi \vdash ?\phi[x/t]$	where t is substitutable for x in ϕ
	[Equality′]	$?\phi \vdash ?((x = y) \wedge \phi)$	where $x \in \text{FV}(\phi)$
	[Const]	$\vdash ?(x = c)$	
	[Equiv]	$!\forall\vec{x}(\phi \leftrightarrow \psi), ?\phi \vdash ?\psi$	where $\text{FV}(\phi) \cup \text{FV}(\psi) = \{\vec{x}\}$
+	[Ref]	$\theta \vdash \theta$	
	[Cut]	If $\Gamma \vdash \theta$ and $\Gamma, \theta \vdash \eta$ then $\Gamma \vdash \eta$	
	[Monotonicity]	If $\Gamma \vdash \theta$ then $\Gamma, \Gamma' \vdash \theta$	
	[Permutation]	If $\Gamma, \theta, \eta, \Gamma' \vdash \theta$ then $\Gamma, \eta, \theta, \Gamma' \vdash \theta$	
+	[Contraction]	If $\Gamma, \theta, \theta \vdash \eta$ then $\Gamma, \theta \vdash \eta$	

Axioms marked with + are derivable

LoI with varying domains predicts the opposite: $?Px \models_{\mathrm{vd}} ?\exists x.Px$ yet $?Px \not\models_{\mathrm{vd}} ?\forall x.Px$, because only a restricted form of universal quantification is allowed. Indeed, once we allow the domains of our first-order models to vary, $\forall x.Px$ does not provide information about the extension of the predicate P, only about the complement of its extension! We could of course dualize the semantics of LoI by consistently interpreting $?\phi$ as asking about the *complement* of the extension of ϕ. In this way, we get the correct empirical predictions, but it remains mysterious why this is the case.

Mathematical Just as LoI can be seen as a logic of isomorphisms, the varying domains version of LoI can be seen as a logic of full partial isomorphisms. Are full partial isomorphisms interesting in their own right? There is at least one reason to believe so: Andréka et al. (1998) introduce a number of fragments of first-order logic, one of which (Fragment F3) can be shown to be the fragment of first order logic invariant under full partial isomorphisms.

Computational The computational perspective provided by Proposition 3.3 becomes even more natural for the varying domain semantics, since it reduces down to the following.

Proposition 3.5 Suppose $!\phi_1, \ldots, !\phi_n, ?\chi_1, \ldots, ?\chi_m \models ?\psi$. Then the extension of ψ in a finite model satisfying $\phi_1, \ldots, \phi_n$ can be effectively computed from the extension of $\chi_1, \ldots, \chi_m$, given the interpretation of all constants. In fact, the computation can be performed in PSPACE.

Note that, unlike in the constant domain case, the domain of the model is not required for the computation. Unfortunately, just as explained for the constant domain case in Section 3.4.3, the converse of Proposition 3.5 does not hold.

Philosophical In response to Tarski's question *What are logical operations?*, the notion of strict developments embodies the view that negation and universal quantification are only logical operations when the body formula is guarded by a restrictor predicate. Semantically speaking, this view submits that the extension of the entire domain itself is not a logical notion. For instance, according to Definition 3.9, bounded (three-part) universal quantification is an acceptable way to build up a strict development, but unbounded (two-part) universal quantification is not. This restriction on the use of universal quantification is reminiscent of the restriction on set comprehension used to avoid Russell's paradox. However, the precise relationship between restricted comprehension and varying domains remains to be worked out.

3.7 Conclusion

We axiomatized Groenendijk's logic of interrogation, and showed that it not only has a natural linguistic interpretation, but also describes equivalence relations between models, reductions among database queries, and logicality of operations. Thus these topics are all related to each other. For example, if we allow the domains of possible worlds to differ, then fewer statements will count as answer to a question, fewer database queries reductions are possible, and fewer operations are logical.

Other interesting variations on LoI may be obtained by replacing rigid individuals by *conceptual covers*, or replacing isomorphisms by *homomorphisms* or *bisimulations*. Using homomorphisms excludes negative answers such as *Britney Spears won't be coming to the party*. Using bisimulations excludes quantitative answers such as *More than 27 people will come to the party*. We leave it as future work to find sound and complete axiomatizations for these cases.

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