Extending modal logic

- What does it mean to “extend” modal logic?
  - Expressive extensions (increasing the expressivity of the language)
  - Axiomatic extensions (restricting the class of frames)
  - Signature extensions (changing the type of structures)

- The challenge: to preserve the good properties of modal logic.
1. Expressive extensions

- We can make the language more expressive. Examples:
  - adding counting modalities,
  - guarded fragment,
  - fixed point operators (modal mu-calculus)

- Trade-off between expressive power and complexity/decidability.
1. Expressive extensions

- One explanation for the good computational behavior of ML:
  - ML is bisimulation invariant, and hence has the tree model property (satisfiability = satisfiability on trees)
  - ML is contained in Monadic Second Order Logic (MSO)
  - MSO satisfiability is decidable on trees (Rabin), in fact even on structures of bounded treewidth (Courcelle).

- This suggests some extensions of ML:
  - Modal mu-calculus is bisimulation invariant fragment of MSO.
  - Guarded fragment has small treewidth model property.
1. Expressive extensions

- For many extensions of ML, analogues of the usual results (axiomatizations, decidability / complexity, frame definability, etc.) have been mapped out.

- There are also some results of a more general nature. Example:
  
  - Modal Lindström Theorem (Van Benthem ‘07): Every proper extension of the modal language lacks either Compactness or bisimulation invariance.
2. Axiomatic extensions

- Often in applications of modal logic, we want to reason about a restricted class of structures, e.g,
  - linear orders (flows of line),
  - finite trees (XML documents or linguistic parse trees),
  - equivalence relations (epistemic logic)

- This means adding axioms to the logic.

- The good properties of the basic modal logic may or may not survive.
2. Axiomatic extensions

- **Theorem (Sahlqvist):** If a frame class $K$ is definable by Sahlqvist formulas, then a complete axiomatization of the modal logic of $K$ is obtained by adding these formulas as axioms to the basic modal logic.

- **Theorem (Bull-Fine-Hemaspaandra):** For every class $K$ of reflexive transitive linear frames, the modal logic of $K$ is finitely axiomatizable, has the finite model property, and is coNP-complete.
We will discuss *frame definability* at length: which properties of frames are definable by modal formulas?

**Theorem**: for frame classes $K$ defined by universal Horn conditions $\forall x_1...x_n(\varphi_1 \land ... \land \varphi_k \rightarrow \psi)$ the following are equivalent:

1. $K$ is modally definable
2. $K$ is closed under bounded morphic images and disjoint unions
3. The Horn conditions can be written so that their left hand sides are tree-shaped.
4. $K$ is definable by Sahlqvist formulas
Frame definability

- Transitivity $\forall xyz(Rxy \land Ryz \rightarrow Rxz)$ is modally definable:

  \[ \begin{array}{c}
  Z \\
  y \rightarrow Rxz \\
  x
  \end{array} \]

- Anti-symmetry $\forall xy(Rxy \land Ryx \rightarrow x=y)$ is not modally definable:

  \[ \begin{array}{c}
  y \\
  \leftrightarrow x
  \end{array} \rightarrow x=y \]
3. Signature extensions

- Sometimes, we want to describe more general type of mathematical structures. Examples:
  - more than one modality ("poly-modal" as opposed to "uni-modal")
  - $k$-ary modalities for $k>1$ (with $k+1$-ary accessibility relations)
  - topological spaces
- Many results in modal logic generalize.
- Occasionally, there can be some surprising differences...
3. Signature extensions

- **Frame satisfiability problem**: given a modal formula, is there a frame on which the formula is valid?

- **Decidable** for uni-modal formula (in fact coNP-complete).
  - If a modal formula is valid on any frame then it is valid on a frame with only one world (Makinson's theorem).
  - Easy to test whether a given frame validates a modal formula.

- **Highly undecidable** for poly-modal formulas (in fact as undecidable as full second-order logic).
3. Signature extensions

- Proof of Makinson’s theorem
  - Let $F_{\text{irr}}$ be the irreflexive one world frame and $F_{\text{refl}}$ the reflexive one.
  - Suppose $\varphi$ is valid on some frame $F$.
  - If $F$ contains a world without successors, then $F_{\text{irr}}$ is (isomorphic to) a generated subframe of $\varphi$, hence $\varphi$ is valid on $F_{\text{irr}}$.
  - If $F$ contains no world without successors, then $F_{\text{refl}}$ is a bounded morphic image of $F$, hence $\varphi$ is valid on $F_{\text{refl}}$. 

3. Signature extensions

- Co-algebras provide a framework for studying signature extensions.
Co-algebras

- Recall that a Kripke model is a structure of the form

\[ M = (D, R, V) \] with \( R \subseteq D \times D \) and \( V : D \to \wp(\text{PROP}) \).

- Equivalently, \( M = (D, f) \) where \( f : D \to \wp(D) \times \wp(\text{PROP}) \).

- Generalizing from this, let \( \tau(X) \) be any term generated by the following inductive definition:

\[ \tau(X) ::= X \mid A \mid \tau + \tau \mid \tau \times \tau \mid \tau^A \mid \wp(\tau) \] with \( A \) any set (e.g., PROP).

- Each such \( \tau \) gives rise to a functor on Set, called a Kripke polynomial functor (KPF).

- A “\( \tau \)-coalgebra” is a pair \( M = (D, f) \) with \( f : D \to \tau(D) \).
Co-algebras

- Kripke models are just co-algebras of a particular functor.
- Other examples: ternary Kripke models, bi-modal Kripke models, ..
- Further generalizations of the class of KPFs are possible, covering also, e.g., neighborhood models.
- With each KPF τ we can associate a “basic modal language”.
- Various results for modal logic (e.g., decidability, finite axiomatization, Goldblatt-Thomason theorem) generalize to arbitrary KPFs. [Rößiger; Jacobs; Kurz & Rosicky]
Topological semantics of ML

- We will discuss in some detail the topological semantics of ML.
  - Take any topological space, say the real line
  - Interpret the $\Diamond$ modality as closure: the closure of a set of real number is obtained by adding limit points.
  - For instance if $X=\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$, then $\Diamond X = X \cup \{0\}$. 
Cross-overs

- There are many cross inter-relations between the three types of extensions of ML. For example,
Outline of the course

- **Tuesday**: expressive extensions
  - Extended modal languages (mu-calculus, guarded fragment, ...)
  - Characterizing languages (bisimulations, Lindström theorems, ...)

- **Wednesday**: axiomatic extensions
  - Frame definability

- **Thursday**: signature extensions
  - Case study: topological semantics

- **Friday**: mixing it all together
  - Properties of topological spaces definable in extended modal languages and/or special requests.