Topological semantics of modal logic

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Outline

1. Introduction to topological semantics
2. Hybrid language on topological spaces
Best known example of a topological space: the real line.

Consider the set $S = \{1, \frac{1}{2}, \frac{1}{4}, \ldots \}$.

- $S$ approximates 0 arbitrarily closely, but $0 \not\in S$.
- Topologists say “$S$ is not closed”.

Every set $X$ has a smallest closed superset, its closure $\mathcal{C}(X)$.

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Some facts about $c$:

1. $X \subseteq c(X)$
2. $c(c(X)) = c(X)$
3. $c(\emptyset) = \emptyset$
4. $c(X \cup Y) = c(X) \cup c(Y)$
Some facts about $\mathcal{C}$:

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Proof of (4)

Suppose $X \cup Y$ approximates $r$.

By the Pigeon Hole Principle, either $X$ or $Y$ approximates $r$. 
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**Topological interpretation of modal operators**

- Proposition letters denote subsets of \( \mathbb{R} \).
- \( \lor, \land \) and \( \neg \) express union, intersection and complement.
- \( \Diamond \) is interpreted as closure (and \( \Box \) as interior).
Some facts about $\mathcal{C}$:

1. $X \subseteq \mathcal{C}(X)$  \hspace{1cm} p \rightarrow \Diamond p$
2. $\mathcal{C}(\mathcal{C}(X)) = \mathcal{C}(X)$ \hspace{1cm} \Diamond \Diamond p \leftrightarrow \Diamond p$
3. $\mathcal{C}([],) = []$ \hspace{1cm} \Diamond \bot \leftrightarrow \bot$
4. $\mathcal{C}(X \cup Y) = \mathcal{C}(X) \cup \mathcal{C}(Y)$ \hspace{1cm} \Diamond (p \lor q) \leftrightarrow \Diamond p \lor \Diamond q$

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Some facts about $\mathcal{C}$:

(1) $X \subseteq \mathcal{C}(X)$ \quad p \rightarrow \lozenge p
(2) $\mathcal{C}(\mathcal{C}(X)) = \mathcal{C}(X)$ \quad \lozenge \lozenge p \iff \lozenge p
(3) $\mathcal{C}(\emptyset) = \emptyset$ \quad \lozenge \bot \iff \bot
(4) $\mathcal{C}(X \cup Y) = \mathcal{C}(X) \cup \mathcal{C}(Y)$ \quad \lozenge(p \lor q) \iff \lozenge p \lor \lozenge q

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$p \rightarrow \diamond p$

$\diamond \diamond p \leftrightarrow \diamond p$

$\diamond \bot \leftrightarrow \bot$

$\diamond (p \vee q) \leftrightarrow \diamond p \vee \diamond q$

Topological interpretation of modal operators

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- $\vee$, $\wedge$ and $\neg$ express union, intersection and complement.
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This interpretation works on any topological space, not only $\mathbb{R}$!
A topological space consists of a set \( D \) (the domain), plus a set of subsets \( C \subset \wp(D) \) (the closed sets), such that

- \( \emptyset, X \in C \)
- If \( X, Y \in C \) then \( X \cup Y \in C \)
  (holds for \( \mathbb{R} \) by the pigeon hole principle, as we saw)
- \( C \) is closed under intersection: for all \( A \subseteq C \), \( \bigcap A \in C \)
  (guarantees that every set has a unique closure).

The closure of a set \( X \) is the smallest \( X' \in C \) with \( X \subseteq X' \).
Equivalently, \( \mathcal{C}(X) = \bigcap \{ X' \in C \mid X \subseteq X' \} \).
**Fact 1:** Topological spaces generalize $\text{S4}$-frames (a.k.a. quasi-ordered sets):

- Every $\text{S4}$-frame defines a topological space over the same domain. The closed sets are the sets closed under taking predecessors.
- Not every topological space is representable by an $\text{S4}$-frame this way. Those that are, are called Alexandroff spaces.

**Fact 2:** Every topological space gives rise to a normal modal logic.
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Theorem (McKinsey-Tarski)

S4 is the modal logic of

- \( \mathbb{R} \),
- any other metric, separable, dense-in-itself space,
- all topological spaces.

A beautiful result, but also disappointing:

- Modal logic is too weak to see interesting properties of \( \mathbb{R} \).
- Richer languages needed to capture interesting topological reasoning.
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Let's now consider the hybrid language $\mathcal{H}(E)$, which has nominals and the global modality.

$$\phi ::= p \mid i \mid \neg \phi \mid \phi \lor \psi \mid \lozenge \phi \mid E\phi$$

- Nominals are proposition letters denoting singleton sets.
- Global modality allows us to say that a formula holds somewhere.

Shorthands:

$$\Box \phi \text{ for } \neg \lozenge \neg \phi \quad A\phi \text{ for } \neg E \neg \phi \quad @i\phi \text{ for } E (i \land \phi)$$

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Example of a hybrid formula valid on $\mathbb{R}$:

$$\Diamond i \leftrightarrow i$$

It says: “the closure of any singleton set is the set itself”, or, equivalently, “every singleton set is closed”.

Not every topological space has this property (which is known as $T_1$-separation).

Corollary

*The hybrid logic of $\mathbb{R}$ is not the hybrid logic of all spaces.*
$T_1$ is definable with nominals

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