Vortex Core Detection: Back to Basics

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ABSTRACT

Analyzing vortices in fluid flows is an important and extensively studied problem. Visualization methods are an important tool, and vortex cores, including vortex-core axes, are frequently objects for which visualization is attempted. A robust definition of vortex-core axis has eluded researchers for a decade. This paper reviews the criteria described in some early papers, as well as recent papers that concentrate on issues of unsteady flows, and attempts to build on their ideas. In particular, researchers have proposed criteria that are desirable for a vortex-core axis that correspond to nonlocal properties, yet current extraction methods are all based on local properties.

Analysis is presented to support the thesis that inaccuracies observed in some popular early methods are due to a mixture of frequencies in the flow field in vortical regions. Such mixtures occur in steady flows, as well as unsteady (time-varying) flows. Thus, the fact that the flows are unsteady is not necessarily the primary reason for inaccuracies recently observed in vortex analysis of such flows. It is hypothesized that time-varying (unsteady) flows tend to be more complex, hence tend to have mixed frequencies more often than steady flows. We further conjecture that an "effective" lack of Galilean invariance may occur in steady or unsteady flows, due to the interaction of low frequencies with high frequencies.

Keywords: vortex core, fluid flow, parallel vectors

1. INTRODUCTION

Analyzing vortices in fluid flows is an important and extensively studied problem. Visualization methods are an important tool, and vortex cores, including vortex-core axes, are frequently objects for which visualization is attempted. However, despite decades of study, there is not yet a commonly accepted *definition* of vortex core or vortex-core axis. This paper attempts to make progress in this direction.

Vortex-core detection and visualization are subjects of several recent PhD theses. Particularly relevant to this paper are the theses of Raphael Fuchs[1] and Jan Sahner.[7] They both contain thorough surveys of the state of the art, as well as substantial original contributions. This paper cites only works that are immediately relevant to its content, and the reader is referred to these theses for additional bibliography.

As the title suggests, this paper attempts to make progress on arriving at a suitable *rigorous* and *physically supported* definition of *vortex core* and *vortex-core axis*. There are many heuristic definitions proposed and empirically evaluated in the literature. Prominent among these is the physically motivated definition of a vortex core *region* proposed by Jeong and Hussain,[3] often referred to as the "λ_2 method." See also Schafhitzel *et al.*[9]

This paper shows, through analysis of examples, that some difficulties in vortex-core axis detection that have been attributed to unsteady flows can also arise in steady flows. Although a new heuristic for vortex-core axis detection is proposed, the main conclusion is that a physically justified definition for the vortex-core axis involves non-local properties of the flow. No "final solution" is proposed, but we hope this paper's analysis helps other researchers make further progress.

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2. BACKGROUND

One of the early papers, if not the first paper, on vortex core detection by numerical methods was by Levy, Degani and Seginer.[4] This is our starting point. This paper deserves careful reading because its development is closely tied to the physics literature on 3D vortical fluid flows. We make several observations about this paper.

1. They define *vortex core* as a region and *vortex-core axis* as a curve within this region.
2. They define the *vortex-core axis* to be a streamline. However, since only steady flows are discussed, it is not clear whether they would define it to be a pathline or something else in an unsteady flow.
3. Although they are cited for proposing that the vortex-core axis is a curve where the vorticity (curl of velocity) is parallel to the velocity, they actually use the phrase "tends to align" and choose a seed point where the angle between vorticity (possibly negated) and velocity is locally minimized. (Van Gelder and Pang present evidence that vorticity is not necessarily parallel to velocity on the vortex-core axis.[11]) Normalized helicity is the cosine of that angle, and is a signed scalar. The vortex-core axis is generated from this seed point. They integrate upstream (backwards in time) from this seed point, but do not continually check the normalized helicity. Rather, they follow the streamline.
4. They observe, both analytically and experimentally, that streamline curvature tends to be lower where absolute normalized helicity is larger. At least near the seed point, the vortex-core axis is the streamline with minimum curvature in the neighborhood. If this curvature were zero, then vorticity and velocity would actually be parallel. However, this is not what they observed numerically.
5. They state that identifying locations where the angle between vorticity and velocity is minimized in the more complicated upstream portions of the flow would be unreliable and very difficult, especially near a critical point from which some vortices emanate.

Another early paper on detection of a vortex-core axis was by Sujudi and Haimes.[10] This paper is widely cited in the visualization literature. Their key idea is to define *reduced velocity* in terms of the real eigenvector of the rate-of-deformation tensor, when that tensor has one real eigenvalue and a pair of complex conjugate eigenvalues. The rate-of-deformation tensor is also known as the gradient of the velocity, and we shall denote it as $\nabla\mathbf{v}$ to avoid verbosity. The reduced velocity is the component of the velocity that is orthogonal to the real eigenvector. Although they consider only steady flows, they observe that $\nabla\mathbf{v}$, and consequently, its real eigenvector are Galilean invariant, that is, they are invariant with respect to any inertial frame of reference.[*] Later researchers observed that $\nabla\mathbf{v}$ is closely related to the *acceleration*, which is given by $(\nabla\mathbf{v})\,\mathbf{v}$ for steady flows, and is itself Galilean invariant.

It is stated several places in the literature that zero reduced velocity is "equivalent" to the acceleration and velocity being parallel. This statement requires some additional conditions to be precise.[5] It is true provided that the acceleration is nonzero (implying that velocity is nonzero), because then they are eigenvectors of $\nabla\mathbf{v}$.

If the real eigenvalue of $\nabla\mathbf{v}$ is zero, then acceleration is zero whenever the velocity is in the eigenvector direction. The convention commonly used is that the zero vector is parallel to everything. It can be shown that this works correctly for a steady pure helical flow. However, we show later in this paper that it shifts the axis when the flow is composed of *two* helical flows with different swirling frequencies.

Researchers working with unsteady flows noted that the lack of Galilean invariance of the above two methods poses problems that are not present in steady flows. This has spurred a search for Galilean invariant criteria for identification of the vortex-core axis. Sahner *et al.*[8] propose several heuristic alternatives that are Galilean invariant, but the curves produced may not have the properties usually associated with the vortex-core axis. See Sahner's thesis for more details.[7]

Fuchs *et al.*[2] analyze some synthetic examples, propose modifications of the basic Sujudi-Haimes method for unsteady flows, and demonstrate improvements on simulation data, as well as the synthetic examples. This paper builds upon that work. In particular, we adopt their criterion that the appropriate vortex-core axis has the simplest geometry among pathlines in its neighborhood.

[*] That is, a different coordinate system moving at a constant velocity with respect to the coordinate system in use.

3. NOTATION AND TERMINOLOGY

Let us introduce some terminology and notation, with which to discuss the technical issues. Our vectors are thought of as column vectors and denoted with boldface lowercase; e.g., $\mathbf{v}$ is usually a 3D velocity that depends on position $\mathbf{x}$ in 3D space and possibly time t. Superscript T on a vector or matrix denotes transpose. Sometimes $\mathbf{x}$ is broken down into $[x, y, z]^T$, but $\mathbf{x}_1$ denotes a particular vector, not the first component of $\mathbf{x}$. Boldface $\mathbf{0}$ denotes a zero matrix or vector of the size required by the context.

The gradient operator ∇ denotes $[\partial/\partial x, \partial/\partial y, \partial/\partial z]$, while $\partial/\partial t$ is written separately. When applied to a column vector, ∇ produces a matrix; thus $\nabla\mathbf{v}$ denotes the rate-of-deformation tensor. Matrices are usually boldface capitals.

We recall that *pathlines* result from integration of $\mathbf{v}$ over $(\mathbf{x}, t)$, and are 1D curves in 4D, whereas *streamlines* result from integration of $\mathbf{v}$ over $\mathbf{x}$, with t held constant, so they are 1D curves in 3D. Either type of line can be used to generate 2D point sets in 4D.

The term *acceleration* is used in more than one way in the literature. For this paper, following Fuchs *et al.*,[2] we say *particle acceleration*, denoted $\mathbf{a}_t$, in the context of an unsteady flow to mean the total time derivative of $\mathbf{v}$. Similarly, the *total jerk vector* is denoted $\mathbf{b}_t$. The equations are:

$$\mathbf{a}_t \;=\; (\nabla\mathbf{v})\,\mathbf{v} + \partial\mathbf{v}/\partial t; \qquad\qquad \mathbf{b}_t \;=\; (\nabla\mathbf{a}_t)\,\mathbf{v} + \partial\mathbf{a}_t/\partial t. \tag{1}$$

However, *acceleration*, denoted $\mathbf{a}_s$, omits $\partial\mathbf{v}/\partial t$ and the steady jerk vector, denoted as $\mathbf{b}_s$, omits $\mathbf{a}_t/\partial t$. For a steady flow, there is no difference.

4. CLASSICAL METHODS SHIFT IN STEADY FLOWS

This paper's first contribution is to show that one phenomenon attributed to unsteady flows and a lack of Galilean invariance is actually due to a different cause. Fuchs *et al.*[2] consider an unsteady flow, described as "rotating vortex rope" (see Figure 1). See Fuchs' thesis for a detailed discussion.[1] Let r, s, k, and ω be scalar constants with k and ω significantly less than s. The unsteady velocity field is given by the equations on the left side of Figure 1. The problem is to identify the points on a vortex-core axis.

For whatever reason, Fuchs *et al.* do not give an explicit expression for the pathlines, but discuss their properties in detail. Table 1 shows the solution pathlines for particles released at $t = 0$ from $\mathbf{x}_0 = [x_0, y_0, z_0]^T$, with t as parameter. They were found (after substituting $z_0 + t$ for z) by standard ODE methods and some trigonometric identities, and may be verified by differentiation. The most significant feature of the equations in Table 1 is that they are the sums of low-frequency terms, involving $((k + \omega)\,t)$, and high-frequency terms, involving $(s\,t)$. To obtain the simplest geometry, the coefficients of the high-frequency terms should be zero. By this criterion, particles released in the plane z_0 at $t = 0$ are on a vortex-core axis if

$$x_0 \;=\; \frac{r\,s}{s - (k + \omega)}\,\cos(k z_0) \qquad\qquad y_0 \;=\; \frac{r\,s}{s - (k + \omega)}\,\sin(k z_0) \tag{2}$$

$$\mathbf{v} \;=\; \begin{bmatrix} -s\,y + & r\,s\,\sin(k\,z + \omega\,t) \\ s\,x - & r\,s\,\cos(k\,z + \omega\,t) \\ & 1 \end{bmatrix}$$

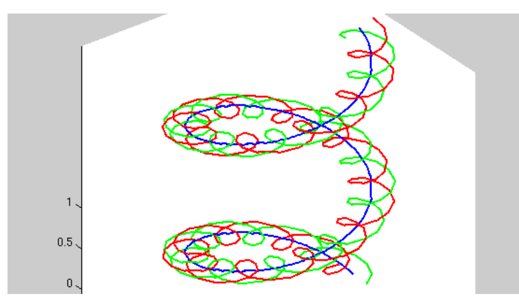

Figure 1. Velocity field (left) and Some streamlines (right) for the rotating vortex rope.

Table 1. Vortex Rope solution. See Figure 1 for the unsteady velocity field.

$$
\begin{aligned}
x &= \frac{r\,s}{s-(k+\omega)}\big(\cos(kz_0)\,\cos((k+\omega)\,t) - \sin(kz_0)\,\sin((k+\omega)\,t) - \cos(kz_0)\,\cos(s\,t) + \sin(kz_0)\,\sin(s\,t)\big) \\
&\quad + \; x_0\,\cos(s\,t) - y_0\,\sin(s\,t) \\
y &= \frac{r\,s}{s-(k+\omega)}\big(\sin(kz_0)\,\cos((k+\omega)\,t) + \cos(kz_0)\,\sin((k+\omega)\,t) - \sin(kz_0)\,\cos(s\,t) - \cos(kz_0)\,\sin(s\,t)\big) \\
&\quad + \; x_0\,\sin(s\,t) + y_0\,\cos(s\,t) \\
z &= z_0 + t
\end{aligned}
$$

For these initial conditions, with some rearrangement of terms, the pathline is

$$
\begin{aligned}
x &= \big(r/\,(1-(k+\omega)/s)\,\big)\,\cos(kz_0 + (k+\omega)\,t) &(3)\\
y &= \big(r/\,(1-(k+\omega)/s)\,\big)\,\sin(kz_0 + (k+\omega)\,t) &(4)\\
z &= z_0 + t &(5)
\end{aligned}
$$

In other words, by this simplest-geometry criterion, each vortex-core-axis pathline traces a helix of radius $r/\,(1-(k+\omega)/s)$ over time (as stated by Fuchs *et al.*, but without an explicit formula).

Fuchs *et al.* also observe that vorticity is parallel to velocity on a helix of radius $r\,\big(1{+}k/(2\,s)\big)$ and acceleration is parallel to velocity on a helix of radius $r\,(1{+}k/s)$. Their first-order improvement on Sujudi-Haimes uses particle acceleration $\mathbf{a}_t$ (which includes $\partial\mathbf{v}/\partial t$) in place of steady acceleration and gives a helix of radius $r\,\big(1{+}(k{+}\omega)/s\big)$. Their second-order improvement on Roth and Peikert[6] uses the total jerk vector $\mathbf{b}_t$ (which includes $\partial\mathbf{a}_t/\partial t$) in place of steady jerk gives a helix of radius $r\,\big(1+(k+\omega)/s+(k+\omega)^2/s^2\big)$.

In all cases, the equations for these helices change only the coefficient in Eqs. 3 and 4, so the curves stay a fixed distance from the "true" vortex-core axis, but these helices are *not* pathlines.

In all four cases, the fact of being parallel is only achieved by the vector other than velocity being zero. The angles *do not* get small in the neighborhood. (These observations are clear without a lot of math because $\mathbf{v}$ has a z component and the higher derivatives do not.) Therefore, the status and quality of these "solutions" is suspect for this reason, besides not being pathlines.

To check further, we tweaked the original flow to have $d^2z/dt^2 = \epsilon\,z$. The critical points in $\mathbf{b}_s$ and $\mathbf{b}_t$ disappear and the parallel loci are well defined. They approach the above-stated values as ϵ approaches zero. Therefore, the solutions appear to be genuine in this case.

We now show that the discrepancies in radius can occur in steady flows, with a very similar flow field. Simply let $\omega = 0$ and $z_0 = 0$ in their example flow field. For particles released from $\mathbf{x}_0 = [x_0, y_0, 0]^T$ in the resulting steady velocity field, the streamlines, with z as parameter, may be found using Table 1, noting that $t = z$.

$$
\begin{aligned}
x &= \frac{r\,s}{s-k}\big(\cos(k\,z) - \cos(s\,z)\big) + x_0\,\cos(s\,z) - y_0\,\sin(s\,z) &(6)\\
y &= \frac{r\,s}{s-k}\big(\sin(k\,z) - \sin(s\,z)\big) + x_0\,\sin(s\,z) - y_0\,\cos(s\,z) &(7)
\end{aligned}
$$

Again, to obtain the simplest geometry, the coefficients of the high-frequency terms, those involving $(s\,z)$, should be zero. (As in the original example, we assume k is significantly less than s.) By this criterion, particles released from $[x_0, y_0, 0]^T$ are on a vortex-core axis if $x_0 = r/(1-(k/s))$ and $y_0 = 0$. For this choice, the streamline comprises a helix of radius $r/(1-(k/s))$, given in terms of parameter z by

$$
x = \frac{r\,s}{s-k}\,\cos(k\,z) \qquad y = \frac{r\,s}{s-k}\,\sin(k\,z) \tag{8}
$$

However, vorticity is parallel to velocity at radius $r\,\big(1{+}k/(2\,s)\big)$, and acceleration is parallel to velocity at radius $r\,(1+k/s)$. The equations are obtained by substituting for the coefficients in Eq. 8, as in the unsteady example.

By these examples we hypothesize that it is the presence of two frequencies, k and s, that throw off the estimates of the traditional methods. The question of steady vs. unsteady seems to be important only to the extent that more complicated flows, with multiple swirling frequencies, may be more common in unsteady flows.

Another observation is that the criterion of "simplest geometry" is not really a local property of streamlines or pathlines. For example, what could be "simpler" than a line with zero curvature? But in both the foregoing examples, the locus of zero curvature is the same as the locus where acceleration (steady) or particle acceleration (unsteady) is parallel to velocity. But this locus is not, itself, a streamline (steady) or a pathline (unsteady). Referring to the steady version of the example, the seed point where curvature is zero is $x_0 = r\,(1+k/s)$, $y_0 = 0$, and the corresponding streamline is

$$x \;=\; \frac{r\,s}{s-k}\cos(k\,z) - \frac{r\,k^2}{s(s-k)}\cos(s\,z) \qquad y \;=\; \frac{r\,s}{s-k}\sin(k\,z) - \frac{r\,k^2}{s(s-k)}\sin(s\,z) \tag{9}$$

A tedious calculation shows that this streamline has zero curvature only when z is a multiple of $\pi/(k+s)$, and therefore does not follow the locus of zero curvature. A similar calculation applies to the unsteady version.

5. A LIMITATION OF EXISTING HIGHER ORDER METHODS

Our next contribution is to show that the proposal by Roth and Peikert[6] to include higher order derivatives in the vortex-axis criteria becomes inaccurate when the vortex-axis (defined by the "simplest geometry" standard) has variable curvature. The proposed extension for unsteady flows by Fuchs *et al.*[2] inherits the same difficulty.

Recall that *jerk vector* is the time derivative of acceleration (which in turn is the time derivative of velocity). Following Fuchs *et al.*, we use $\mathbf{b}_s$ for the steady value and $\mathbf{b}_t$ for the unsteady value. Roth and Peikert proposed that the vortex-core axis should be the locus where the jerk vector $\mathbf{b}_s$ is parallel to the velocity $\mathbf{v}$, to accommodate a bending axis. Fuchs *et al.* proposed that total time derivatives be used for the acceleration and jerk vector in unsteady settings.

$$\mathbf{a}_t \;=\; (\nabla \mathbf{v})\,\mathbf{v} + \partial \mathbf{v}/\partial t \qquad \mathbf{b}_t \;=\; (\nabla \mathbf{a}_t)\,\mathbf{v} + \partial \mathbf{a}_t/\partial t. \tag{10}$$

Later, we present an alternative that generalizes both proposals and handles variable curvature better.

First, we present a synthetic steady example in which the streamline of simplest geometry does not have the property that the jerk vector is parallel to velocity. The main idea is shown in Figure 2, and is similar to the horse-shoe vortex shown in Schafhitzel *et al.*[9] The flow is easy to describe, although the equations are a bit complicated. Some software is available on the Internet.[†]

The velocity field that is the basis for Figure 2 and related discussion is defined as follows. The family of fields can be defined by choosing various frequencies, ω and s as parameters. Normally, $\omega < s$.

$$\begin{aligned} \tau &= \frac{\operatorname{atan2}(y/3,\,x)}{\omega} & L(\tau) &= \sqrt{5 + 4\,\cos(2\,\omega\,\tau)} \\ x_c(\tau) &= \cos(\omega\,\tau) & & \\ y_c(\tau) &= 3\,\sin(\omega\,\tau) & \mathbf{v}(\tau) &= \frac{\omega}{L}\begin{bmatrix} (-y_c/3)\,r \\ 3\,x_c\,r \\ 0 \end{bmatrix} + s\begin{bmatrix} (-x_c)\,z \\ (-y_c)\,z \\ r-1 \end{bmatrix} \\ r(\tau) &= \sqrt{x^2 + (y/3)^2} & & \end{aligned} \tag{11}$$

Note that "atan2" is the two-argument inverse tangent function. This is not a time-varying field because the parameter τ is defined in terms of the particle position. The field has no singularities for $|z| \le \sqrt{x^2 + (y/3)^2}$.

There is a streamline of simplest geometry that traces the ellipse $x^2 + (y/3)^2 = 1$ in the $z = 0$ plane, at constant velocity ω. Other streamlines swirl with angular velocity s in a certain plane around a particle that moves along the simplest streamline. This plane is determined by the particle position and the z-axis.

As shown in the figure, for the simple streamline, the acceleration vector (black) is always orthogonal to the velocity, but its magnitude grows as the curve sharpens. The jerk vector (magneta), being the derivative of

[†] `http://www.cse.ucsc.edu/~avg/Vcore/` offers supplemental material, including `matlab` code to generate these curves and color images.

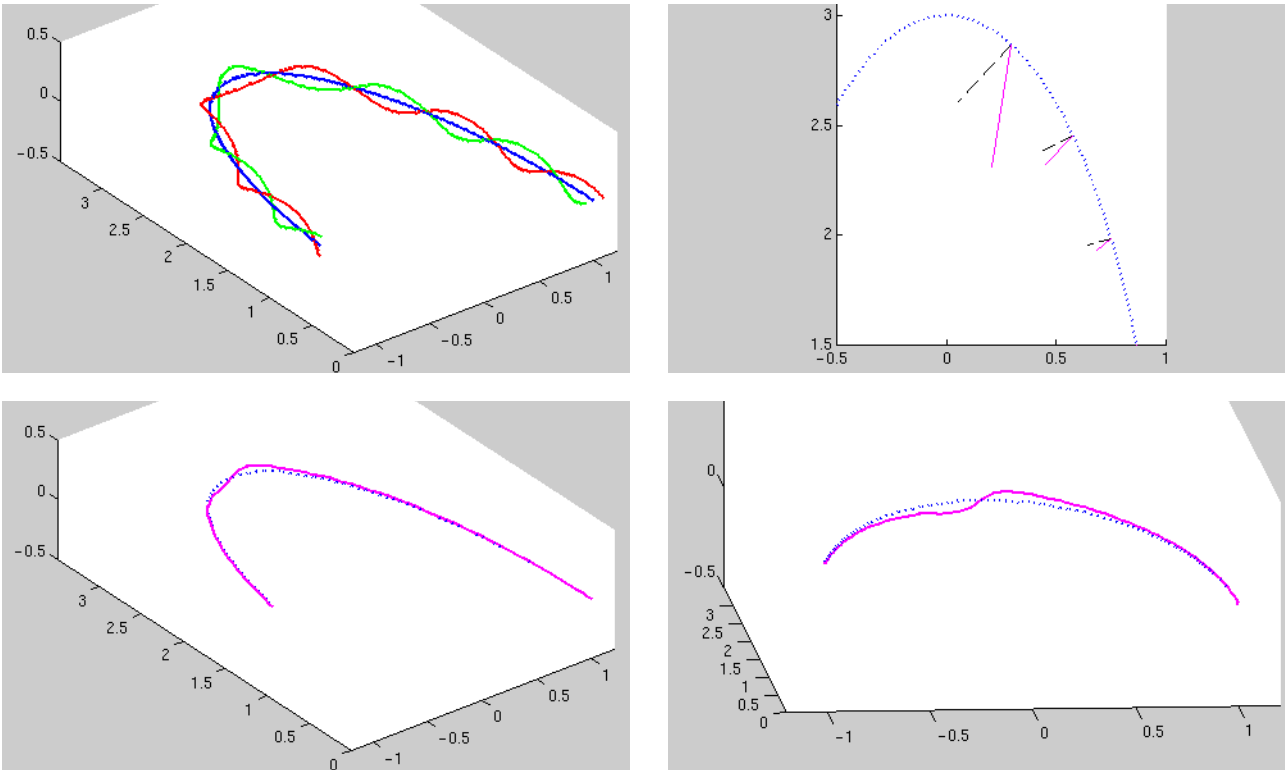

Figure 2. Mixed nonlinear swirling steady flow. *Upper left*: several streamlines, including blue dotted line with simplest geometry; this streamline has constant velocity magnitude, remains in the x-y plane, and appears as blue dotted in subsequent images. *Upper right*: Acceleration (black dashed) increases in magnitude as curve sharpens; jerk (magenta solid) is not parallel to velocity, but both vectors lie in the x-y plane. *Lower left*: Magenta solid line shows locus where jerk *is* parallel to velocity; this does not follow a streamline. *Lower right*: Another view of lower-left image shows the magenta solid line rising, then falling negative, then returning to zero, in z.

the acceleration, has a component parallel to the acceleration, due to its growth in magnitude, and thus is not parallel to the velocity on the simple streamline.

However, there *is* a locus where the jerk vector is parallel to the velocity, which starts out very close to the simple streamline, but veers upward (positive z) as the curvature increases. This may be understood geometrically as moving up far enough to pick up the right amount of offsetting jerk vector from the swirl to cancel the nonparallel component produced by the simple streamline. After passing the ellipse major axis, the curvature is decreasing, and this locus veers downward into negative z.

To support the above claim, it can be shown by calculus or by geometric reasoning that on any ellipse defined by $x^2 + (y/3)^2 = 1$, $z =$ constant (i.e., directly above the simplest streamline, with z being "up"), the vectors $\mathbf{v}$ and $\mathbf{b}$ have no z component. Then, given a point $(x, y, 0)$ on the simplest streamline, the required "elevation" z to make $\mathbf{v}(x, y, z)$ and $\mathbf{b}(x, y, z)$ parallel can be found by solving an equation in z, with coefficients given by various cross-products. Details are in matlab code in the Internet supplement mentioned above.

6. A NEW VORTEX-CORE AXIS METRIC

The Sujudi-Haimes criterion, reformulated, amounts to the requirement that the angle between the acceleration and the velocity should be as small as possible. Fuchs *et al.* advocate using particle acceleration and velocity for unsteady flows, but the idea is the same. If the angle between these two vectors can remain at zero while moving along the pathline, that pathline clearly qualifies as the vortex-core axis. As the examples showed, it may be the case that no streamline satisfies this condition over an extent in space.

Geometric analysis suggests that a streamline is simplest when it remains in a single plane, or at least the plane changes as slowly as possible. We propose that the metric for flows too complicated for the Sujudi-Haimes criterion should be the *triple scalar product* of velocity, acceleration, and jerk. For unsteady flows, use velocity, particle acceleration, and total jerk. The triple scalar product is simply the determinant of the matrix formed with these three vectors as columns. If this quantity is zero, the three vectors over-determine a plane. For consistent scaling, divide the result by $|\mathbf{v}|^3$.

The triple scalar product is the natural extension of the idea of parallel vectors to a set of three vectors in the following sense. The (magnitude of the) cross product of two vectors is equal to the *area* of the 2-D parallelogram that they define (let each vector be an edge of the parallelogram). The triple scalar product of three vectors is equal to the *volume* of the 3-D parallelepiped that they define (let each vector be an edge of the parallelepiped).

Two vectors, in general, provide a basis for a 2-D space, but if they are parallel (the cross product is zero), then it degenerates to a 1-D space, which is simpler. Three vectors, in general, provide a basis for a 3-D space, but if they are not all linearly independent (the triple scalar product is zero), then the basis spans a 2-D or 1-D space. Since we are looking for a curve of simple geometry, trying to reduce the dimension of the space that it spans locally is a logical idea.

Recall that the *curvature* of a pathline is defined by $\mathbf{v} \times \mathbf{a}_t / |\mathbf{v}|^3$. Actually, the curvature is usually a scalar quantity that is the magnitude of this vector quantity; this is a natural extension to associate a direction with the curvature. Further, the *torsion* of a pathline is defined by $((\mathbf{v} \times \mathbf{a}_t) \cdot \mathbf{b}_t)/|\mathbf{v} \times \mathbf{a}_t|^2$.[6] Consider the projection of a pathline in a neighborhood $|t - t_1 < \Delta t|$ on a plane in 3D space that is orthogonal to $\mathbf{v}(t_1)$ on that pathline. If $(\mathbf{v} \times \mathbf{a}_t) \cdot \mathbf{b}_t \neq 0$, that projection is curved in the neighborhood of t_1, suggesting swirling around a line that is parallel to, but offset from the tangent to the pathline.

It is interesting to note that this is a signed scalar metric, like the normalized helicity used by Levy *et al.*, and it can detect the orientation of the swirl. Place the vectors in the matrix in the natural order, $\mathbf{v}$, $\mathbf{a}_t$, and $\mathbf{b}_t$. If the determinant is positive the pathline swirls counter-clockwise as it moves toward the observer. The determinant is equal to the triple scalar product, $\mathbf{v} \cdot \mathbf{a}_t \times \mathbf{b}_t$.

Clearly, if *either* the acceleration or the jerk vector is nonzero and parallel to the velocity, the triple scalar product is zero, so our new metric finds all those solutions. However, it may be the case that *no* pathline has this metric equal to zero over an extent in space.

The metric works perfectly on the example in Section 5, being zero on the streamline of simplest geometry. In fact, the proposal grew out of analysis of this kind of example. However, for the "vortex rope" example in Section 4, the situation is more complicated. Without a lot of math, just geometric analysis, using the fact that the z-components of the higher derivatives are zero, and the z-component of the velocity is constant, we are able to determine that the triple scalar product is *constant* on the pathlines of simplest geometry. On nearby pathlines, with a nonzero high-frequency component, the metric rises above and falls below this value with period $2\pi/s$. As mentioned, the triple-scalar-product metric will be zero on the helices of *both* radii determined by parallelism of (particle) acceleration with velocity and parallelism of (total) jerk with velocity.

7. DISCUSSION AND CONCLUSION

We reviewed the principles for defining vortex-core axis espoused by previous researchers, with attention to motivations from physics and geometry. We adopted the idea that pathlines of "simplest geometry" in a vortical region are the best choice for the vortex-core axis, while recalling that the original idea of a vortex-core axis is some pathline (but in a steady flow). In any case it was not conceived to be an arbitrary locus of points that satisfy a local geometric property.

In this sense, there are inherent limitations on how well local properties can identify a vortex-core axis. We believe the approach of Levy *et al.* to start in an uncomplicated part of the flow and work back toward the complicated part along a pathline or streamline deserves renewed investigation.

For unsteady flows the vortex-core axis is actually a two-dimensional set of points, so the situation is much more complicated. Since material moves along pathlines in Navier-Stokes solutions, a family of pathlines seems like the most natural candidate to define a family of vortex-core axes.

In keeping with our belief that nonlocal properties need to be considered to provide a physically convincing solution, a possible approach is to focus on how to choose a set of seed points for pathlines that will define the family of vortex-core axes. Possibly variational methods can be applied to evaluate nonlocal properties.

We analyzed the behavior of current techniques, and presented evidence that some of the "shifts" observed are due to flows composed of a mixture of different frequencies. We showed that these shifts can occur also in steady flows. In particular, the higher-order methods that perform very well when the vortex-core axis bends at a constant rate were shown to shift when the bending rate varies along a streamline, and two frequencies are present in the flow. This phenomenon was exhibited in a steady flow, but obviously anything that happens in a steady flow can also happen in an unsteady flow. Schafhitzel *et al.*[9] evaluate several higher-order methods empirically.

Several researchers have observed that classical geometric techniques for location of a vortex-core axis undergo shifting in unsteady flows, and attribute this to a lack of Galilean invariance. While we have shown that the phenomena occur also in steady flows, we agree that the issue very likely involves the lack of Galilean invariance. In a flow with a high frequency and a low frequency, we believe the low frequency motion approximates the effect of a moving observer, *even though it is not really inertial*. In other words, the low-frequency motion is fairly straight, compared to the high frequency, and it shifts the perception of where the high-frequency core is located.

Finally, we proposed a triple-scalar-product metric to try to capture the benefits of both first-order and second-order techniques, and add some new benefits. We showed one class of flows, involving a vortex-core axis that bends at a variable rate, for which this metric outperforms previous techniques. However, further investigation is needed to determine how well it does in a large variety of flow situations. Quite possibly its best use is in combination with other criteria. In any case, this is yet another local property, and cannot be expected to be a panacea.

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