Input Distance and Lower Bounds for Propositional Resolution Proof Length

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Overview

General Question: What properties of CNF formulas force resolution refutations to be "long"? (CNF Formula: $\mathcal{F}$ is a set of clauses $\{C_i\}$

Derived Clauses: $D = \mathbf{res}(q, D_1, D_2)$ where **res** denotes binary resolution with q as the clashing literal.

- Input clauses, those in $\mathcal{F}$, are considered to be derived in 0 steps.

Clause Width $|D|$: number of literals in D.

- Derivation Width: maximum clause width of any derived clause.

Input Distance $\Delta(D)$: Minimum over $C_i \in \mathcal{F}$ of $|D - C_i|$ (*note asymmetry*).

- Also written $\Delta_F(D)$ where context does not make $\mathcal{F}$ definite.
- Derivation Input Distance: maximum input distance of any derived clause.

New Result: If every refutation of $\mathcal{F}$ has a "large" input distance, then every refutation of $\mathcal{F}$ is "long".

Some Notation

(Resolution) Derivation: A *directed acyclic graph* π in which each derived clause D is represented by a vertex with two arcs to the operands (clauses) used to derive D.

- Input clauses C_i are included, and have no arcs leaving.
- For general resolution, in-degree is arbitrary.
- For tree-like resolution, in-degree is 1 for derived clauses, arbitrary for C_i.

(Closest) Previous Work:

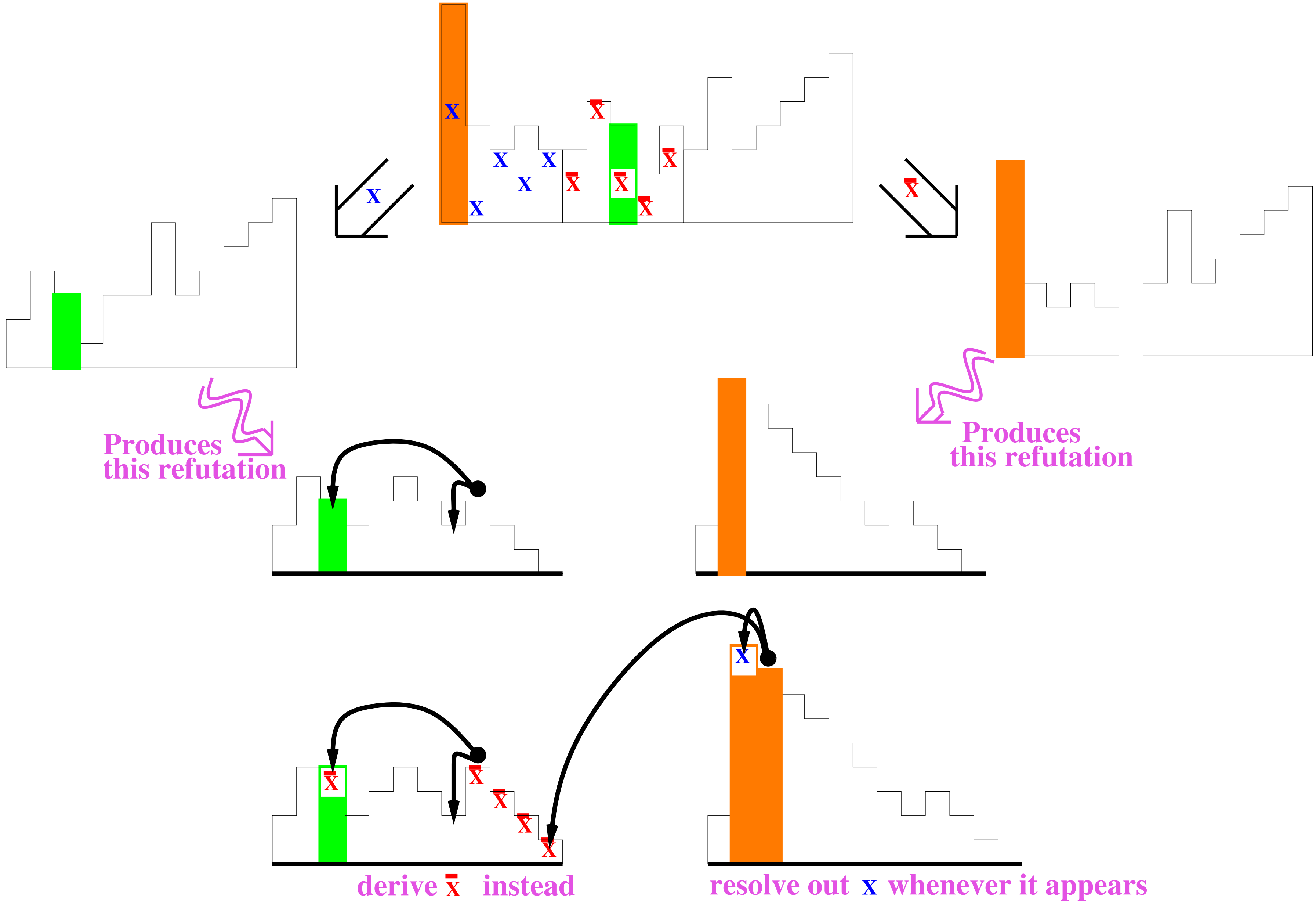
Anderson & Bledsoe, JACM 1970,

Ben-Sasson & Wigderson, JACM 2001,

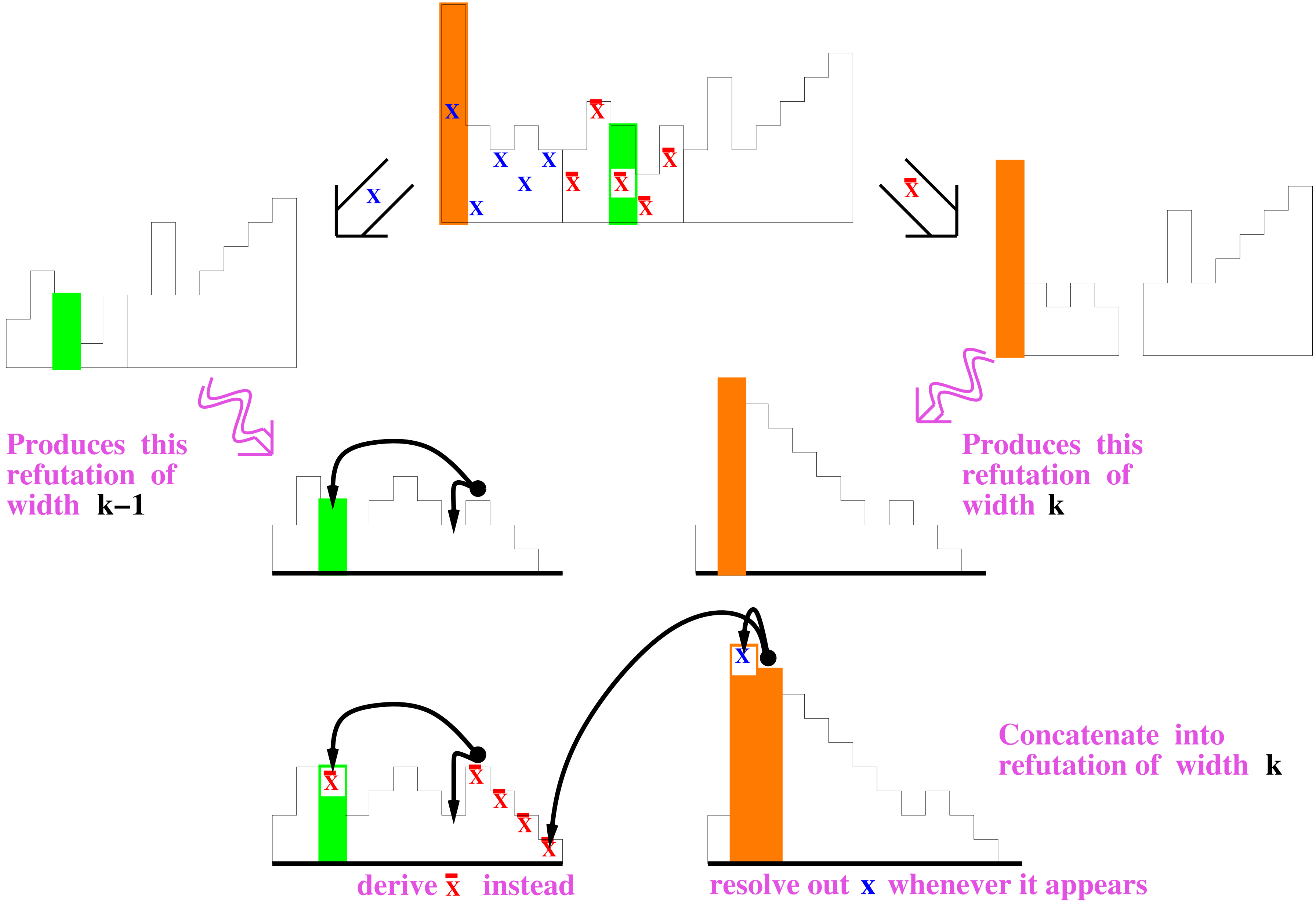
Bonet & Galesi, Computational Complexity 2001.

Anderson & Bledsoe introduced a general framework for proving properties of resolution systems by induction. They gave uniform completeness proofs for strategies: Linear, Set-of-Support, Positive, Hyperresolution and others.

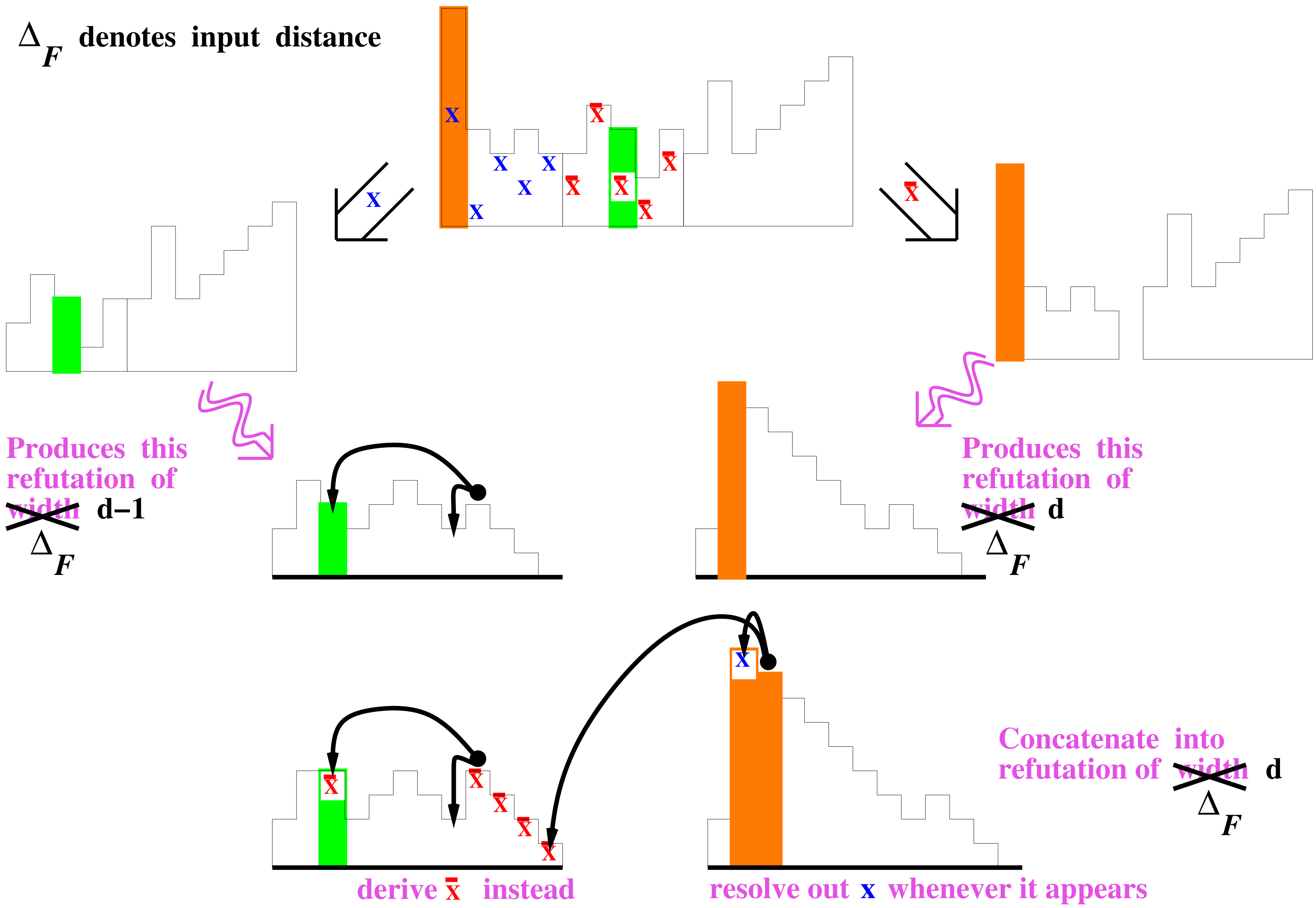
Anderson and Bledsoe's Idea:



Ben-Sasson and Wigderson's Idea (One of Them):



This Paper: Use Input Distance instead of Clause Width



Consequence: “Short” Refutation Implies Refutation with “Small” $\Delta_{\mathcal{F}}$

Tree-Like Resolution: There is refutation π with $\Delta_{\mathcal{F}}(\pi) \leq \lceil \lg S_T \rceil$

General Resolution: There is refutation π with $\Delta_{\mathcal{F}}(\pi) \leq \left\lceil \sqrt{8N \ln S} \right\rceil$

> S_T denotes length of shortest tree-like refutation of $\mathcal{F}$.
>
> S denotes length of shortest general refutation of $\mathcal{F}$.
>
> N denotes number of propositional variables in $\mathcal{F}$.

Notes:

Ben-Sasson and Wigderson have similar formulas, but with incremental width, $(\mathrm{width}(\pi) - \mathrm{width}(\mathcal{F}))$, in place of $\Delta_{\mathcal{F}}(\pi)$.

The proofs follow theirs, just using properties of $\Delta_{\mathcal{F}}(\pi)$.

In both cases, the refutation with “small” $\Delta_{\mathcal{F}}$ or “small” incremental width might be exponentially longer than that with the shortest length.

Ben-Sasson and Wigderson's Idea (Another One)

Let π be any refutation of $\mathcal{F}$.

Use a complexity metric $\mu(D)$ on derived clauses D to show that some derived clause of π has a "large" incremental width.

Useful Technique:

- Define a set of formulas $\mathcal{A} = \{A_i\}$ that express natural constraints of the underlying problem, whose union is $\mathcal{F}$.
- $\mu(D)$ is the minimum number of A_i needed to logically imply derived clause D.

***This paper*:**

Use $\mu(D)$ to show that some derived clause D in π has a "large" $\Delta_{\mathcal{F}}(D)$.

Example: Show Refutations of Pigeon-Hole Formulas Have Large $\Delta_{\mathcal{F}}$

$\mathrm{PHP}(n+1,n)$ encodes statement that $n+1$ pigeons, which form a clique, cannot be colored with n colors. (Generalization: m pigeons.)

- $x_{i,j}$ means pigeon i has color j.
- positive long clauses: Each pigeon has *some* color.
- negative binary clauses: No two pigeons have *same* color.
- $N = n(n+1)$ is the number of propositional variables.

Define $\mathrm{PigeonOK}(p)$ to assert that pigeon p has *some* color and no pigeon adjacent to p (which is all of them) has the *same* color as p.

- The positive long clause for pigeon p;
- All the negative binary clauses that mention p.

$\mu(D)$ is the minimum number of PigeonOK's needed to logically imply D.

If $D = \mathbf{res}(D_1, D_2)$, then $\mu(D) \leq \mu(D_1) + \mu(D_2)$.

In any refutation, some D has $n/3 \leq \mu(D) < 2n/3$.

Refutations of Pigeon-Hole Formulas Have Large $\Delta_{\mathcal{F}}$ (cont.)

Suppose:

- $n/3 \le \mu(D) < 2n/3$ and
- D has fewer than $n/3$ negative literals and
- the positive literals of D mention fewer than $n/3$ pigeons.

We derive a contradiction. Thus $\Delta_{\mathcal{F}}(D) \ge n/3 - 2$.

Let I be any minimal set of pigeons such that $|I|$ defines $\mu(D)$. *That is,*

$$E_0 \stackrel{\text{def}}{=} \Big(\bigwedge_{I} \text{PigeonOK}(i) \quad \wedge \quad \neg(D)\Big) \qquad \text{is unsatisfiable.}$$

Choose pigeon $p \in I$ that is *not* mentioned among positive literals of D.
By minimality of I:

$$E_1 \stackrel{\text{def}}{=} \Big(\bigwedge_{I-\{p\}} \text{PigeonOK}(i) \quad \wedge \quad \neg(D)\Big) \qquad \text{is satisfiable.}$$

We show that E_0 is also satisfiable. ***Contradiction.***

Refutations of Pigeon-Hole Formulas Have Large $\Delta_{\mathcal{F}}$ (cont.)

Proposition: If M is any total assignment and M_1 is obtained from M by swapping assignments for pigeons q and r (that is, $x_{q,j} \mapsto x_{r,j}$ and $x_{r,j} \mapsto x_{q,j}$): then PigeonOK(i) has the same truth value under M and M_1.

Given: $E_1 \stackrel{\text{def}}{=} \left(\bigwedge_{I-\{p\}} \text{PigeonOK}(i) \quad \wedge \quad \neg(D) \right)$ is satisfied by M.

Need to show: $E_0 \stackrel{\text{def}}{=} \left(\bigwedge_{I} \text{PigeonOK}(i) \quad \bigwedge \quad \neg(D) \right)$ is also satisfiable.

M defines a coloring for pigeons in $I - \{p\}$.

M_0: like M but unnecessarily true variables are false.

Some color, say w, is not mentioned in any variable assigned true by M_0.

Choose pigeon u (for unknown) not among positive literals of D and not in I.

M_1: like M_0 but pigeon u is colored w. M_1 satisfies E_1.

M_2: Swap M_1 assignments of pigeons u and p. M_2 satisfies E_0. **QED**

Discussion

"Short Proofs Are Narrow — Resolution Made Simple"

In what sense is this catchy title true? (Not to detract from the many good results)

- On many specific formulas, the shortest known refutation is (often exponentially) shorter than the narrowest, and *much wider*.

  Example: $\mathrm{PHP}(n+1,n)$ can be refuted with clause-width n, but ...
  Cook's exponentially shorter refutation uses width $n^2/4$.

width	length
n	$\Theta(n!)$
$n^2/4$	$\Theta(n^2 2^n)$

- Bonet and Galesi demonstrate that $\mathrm{MGT}(n)$ requires clause-width in $\Omega(n)$, although it is known to have polynomial-length general refutations.

"Optimality of Size-Width Tradeoffs for Resolution"

"Tradeoff" seems more accurate, and implies correctly, that ...

Short Proofs Are Wide.

Conclusion

Input Distance ($\Delta_{\mathcal{F}}$) has been introduced as a refinement to the incremental clause width metric.

Pigeon-Hole Formula refutations require $\Omega(n)$ input distance.

As with clause width on Extended PHP, the $\Omega(n)$ bound becomes $\Omega(\sqrt{N})$.

Based on known theorems, this does not lead to a lower bound for general resolution refutation length.

More study is needed to evaluate, and perhaps further improve, this metric.

Refutations of Pigeon-Hole Formulas Have Large $\Delta_{\mathcal{F}}$ (cont.)

Proposition:

Refutations of Pigeon-Hole Formulas Have Large $\Delta_{\mathcal{F}}$ (cont.)

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