Simulation of Elastic Membranes with Triangulated Spring Meshes UCSC-CRL-97-12

Allen Van Gelder and Jane Wilhelms
avg@cs.ucsc.edu and wilhelms@cs.ucsc.edu
Computer Science Dept.
University of California, Santa Cruz 95064
E-mail: {avg,wilhelms}@cs.ucsc.edu

July 3, 1997

Abstract

Spring meshes have been used to model elastic material by numerous researchers, with skin, textiles, and soft tissue being typical applications. However, given a specified set of elastic material properties, the question of whether a particular spring mesh accurately simulates those properties, has been largely ignored in the literature. In two dimensions, given a discretization of a membrane as a triangle mesh, the standard finite element method analyzes each triangle approximately as a membrane with specified elastic properties, computing stresses and strains. An alternative is to regard each edge as a spring, assuming the springs are connected by "pin-joints" at the vertices of the discretization. This alternative, called a "spring mesh", is computationally more attractive. Previous reports on the technique are silent on the subject of assigning stiffness to the various springs. This paper shows that assigning the same stiffness to all springs badly fails to simulate a uniform elastic membrane, for equilibrium calculations. A formula for spring stiffness that provides a more accurate simulation is then derived. Its accuracy is demonstrated on test and practical mesh examples. It is also shown that an exact simulation is not possible, in general.

KeyWords

Computer graphics, finite element method, spring mesh, elastic membrane, tissue models.

1 Introduction

Discrete models of elastic materials have many applications, particularly in computer graphics. The driving force for this research was the problem of modeling skin so that it would stretch and slide realistically over an underlying body of modeled muscles and bones. Particularly, on animal skin with markings, the markings should shift realistically in response to underlying limb movements, and so on. For this purpose, skin can be regarded as a uniform, isotropic elastic membrane. However, the results apply equally well to non-uniform, isotropic elastic membranes, as well as to 3D isotropic elastic solids. An isotropic elastic material is one whose *local* deformation in response to force is independent of direction; however this response might vary

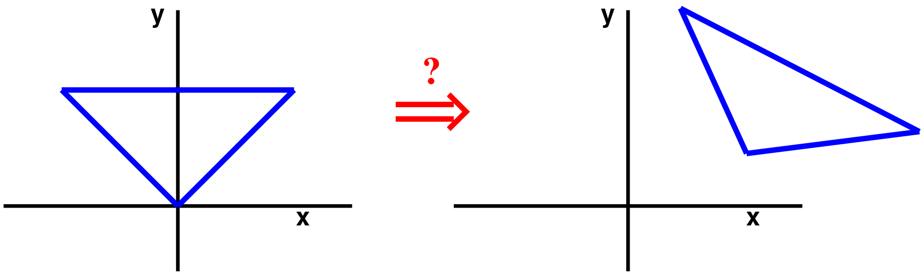

Figure 1: Triangle in rest position at left has arrived at the vertex positions shown at right. The problem is to determine what stresses are now present in the triangle. For the membrane model, this involves decomposing the transformation into a rigid-body motion and a deformation.

over space, in which case the material is non-uniform, but still may be isotropic. We have not investigated non-isotropic elasticity.

Our main concerns were that the model arrive at accurate equilibrium positions within a reasonable amount of computation. The membrane is given as a triangulated surface, produced by some independent surface generation process, so we do not have control over the sizes and shapes of the triangles. The traditional finite element method models each triangle as an individual elastic membrane, with compatibility conditions to ensure they join correctly after deformation. An alternative seen in computer graphics literature is to model each *edge* in the triangulated surface as a spring, i.e., an idealized one-dimensional elastic object. Springs are connected as pin joints at the vertices. This approach offers conceptual simplicity and computational simplicity compared to the first, but it is unclear under what conditions it produces the same results.

To illustrate one reason spring meshes are easier to work with, consider the problem depicted in Figure 1, which might be termed the *inverse elasticity problem*. Suppose we are given that the original triangle at the left is now in the position shown at the right. What internal elastic forces is that triangle producing? This is a completely straightforward problem for the spring-mesh model. For the membrane model, the solution is not at all obvious.

This paper investigates the connection between the two models, with particular attention to finding the spring constants for the second model that allow it to simulate the first model. Previous papers have not spelled out a scheme for setting spring constants, leaving the presumption that they are all equal, or in the case of a multi-level model, that they are all equal within each level. Several test cases demonstrate that this choice of making all spring constants equal produces quite noticeable distortions (see Section 7). We show that an exact simulation between the two models is impossible in general (see Example 5.1). However, we derive a formula for spring constants that enables an irregular spring mesh to simulate uniform elasticity more accurately (see Section 6). Because the formula is local, it extends immediately to the case of non-uniform membranes. Tests show that the approximate formula produces very accurate results in most cases. Meshing strategies to avoid the worst case are discussed.

2 Background

Although a rich literature exists on finite element methods for engineering elasticity, this literature has considered mainly materials that undergo very little deformation within their elastic limits, such as metals, and is more concerned with distribution of forces, rather than the deformation itself [TG51, Whi85, ZT89]. Computer graphics applications are typically more concerned with observable deformations of less rigid materials, such as skin, other soft tissue, fabrics, inflated balls, etc. [TPBF87, TF88, TW88, TW90, VCMT95, LTW95, KGC+96].

Terzopoulos *et al.* introduced the use of elasticity theory for modeling deformable materials into the computer graphics literature [TPBF87]. The work was extended to inelastic materials [TF88] and to combinations of rigid and deformable materials [TW88] the next year. Terzopoulos and co-authors have more recently applied elastic models to simulation of the face [TW90, TW91, LTW95] and fish [TTG95]. They use a layered model of facial tissue consisting of triangular prism finite elements connected by biphasic springs.

Miller used a simple elastic spring-mesh model with fixed spring constants to model the motion of snakes and worms [Mil88]. Chen and Zeltzer used a finite element model to simulate a skeletal muscle [CZ92]. Isometric brick elements were used.

Celniker and Gossard [CG91] used finite elements for general free-form modeling. Primitives automatically deformed to minimize energy based on user-supplied values. Curve and surface elements were used.

Volino *et al.* used a finite element model to simulate deformable surfaces such as cloth [VCMT95]. Although their model seems related to this paper, the details are insufficient for an exact comparison. Also, there appear to be some typographical errors in their formula, because applying it literally to an isosceles right triangle gave strains in the wrong direction.

Koch *et al.* recently described the use of finite element models for simulating facial surgery. They process photogrammetric and CT scan data of the face to create a finite element model of the facial surface and soft tissue. Stiffness parameters depend on the type of material modeled: bone, skin, muscle, or fat. A globally C1 continuous finite element model with nonlinear shape functions, based on triangular prism elements, is developed. The system is built on commercial available modeling and animation systems for interactive geometric manipulation and rendering. The computation of the global stiffness matrix took 17 minutes, and solving the global equation system took 11 minutes on an SGI Indigo 2.

In the above cited research, elastic materials are represented as spring meshes in several cases. These papers either explicitly state that the stiffness coefficients of these springs are independent of the geometry (but often depend on the material), or they imply it by not giving any method to choose the values. As we shall show experimentally, for a membrane of uniform material, identical spring stiffness coefficients for all edges in the mesh lead to noticeable distortions. This paper derives a geometrically based formula for varying the coefficients to provide a more accurate simulation of an isotropic membrane (Section 6). Again, experimental results confirm the accuracy of this formula on test meshes, as well as practical meshes (Section 7).

3 Isotropic Elastic Materials

For an isotropic, linearly elastic material, the relationship between stress and strain depends on two parameters of elasticity, called Young's modulus (E) and the Poisson coefficient (ν) [TG51]. These parameters may vary by position in the material, but if they do not we say the material is *uniform*. For a membrane of constant thickness t, we write $E_2 = Et$, so that E_2 is the "two-dimensional Young's modulus". Similarly for a one-dimensional spring (also called truss), of constant cross-sectional area a, we write $E_1 = Ea$.

Recall that *stress* is the force per unit length due to deformation of the material, while *strain* is deformation per unit length, with deformation being the local change in length in a particular direction. Rigid body motions consisting of translations and rotations do not contribute to deformation.

Young's modulus is also called the *modulus of stiffness*, because it measures the material's resistance to deformation. The Poisson coefficient ν measures the tendency of the material to *shrink* in directions orthogonal to a stretching stress, or *expand* in directions orthogonal to a compression stress. All natural materials have $\nu \geq 0$. The assumption of incompressibility implies $\nu = 0.50$. Timoshenko and Goodier recommend using $\nu = 0.25$ for "most materials" [TG51].

4 Modeling Elastic Membranes

In elasticity terminology, a *membrane* is an idealized two-dimensional elastic material for which forces needed to bend the surface are negligible in comparison to those needed to stretch or compress it. The simplest case is a planar membrane. In most cases the finite element method approximates a nonplanar membrane by a collection of planar patches, although higher order surfaces are sometimes used [Whi85, Fen86]. We shall restrict attention to the case of planar triangular patches. To make the paper accessible to a wider audience, and to provide geometrical insights, we shall avoid the use of tensor notation in the development.

Consider first a planar surface with a given discretization in the form of a set of vertices

$$V = \{v_i,\, i = 1, \ldots, n\}\,,$$

a triangulated planar graph connecting those vertices, producing a set of nonoverlapping triangles,

$$\{T_e,\, e = 1, \ldots, m\}\,.$$

We make no assumption about the regularity of the triangulation. We assume the locations of the vertices in E^2 (Euclidean 2-space) are given for the situation in which there are no stresses.

In any orthonormal coordinate system (u, v), denoting stress components by σ_{uu}, σ_{vv}, and σ_{uv}, and denoting strain components by ε_{uu}, ε_{vv}, and ε_{uv}, we have

$$\begin{aligned} \varepsilon_{uu} &= (\sigma_{uu} - \nu\sigma_{vv})/E_2 && (1)\\ \varepsilon_{vv} &= (-\nu\sigma_{uu} + \sigma_{vv})/E_2 && (2)\\ \tfrac{1}{2}\varepsilon_{uv} &= (1+\nu)\sigma_{uv}/E_2 && (3) \end{aligned}$$

where E_2 and ν were defined in Section 3. We avoid a matrix notation here, although it is seen in some engineering texts [BR75, Fen86], because the three components of σ and of ε are more appropriately viewed as 2×2 symmetric matrices, rather than 3-vectors.

Suppose coordinate frame (u, v) is rotated counter-clockwise from the coordinate frame (x, y) by angle θ. Let $R(\theta)$ denote the rotation matrix:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{4}$$

so that $\binom{u}{v} = R(\theta)\binom{x}{y}$. Then stress matrices in different coordinate frames are related through the similarity transformation:

$$\begin{bmatrix} \sigma_{uu} & \sigma_{uv} \\ \sigma_{uv} & \sigma_{vv} \end{bmatrix} = R^{-1}(\theta) \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} R(\theta) \tag{5}$$

For symmetric matrices, as is well known, there are two choices of θ that diagonalize the left-hand side, and are 90 degrees apart. These are called the principal directions for the stress. Similar remarks apply to the strain matrix, but note the factor of $\frac{1}{2}$ on the off-diagonal elements:

$$\begin{bmatrix} \varepsilon_{uu} & \frac{1}{2}\varepsilon_{uv} \\ \frac{1}{2}\varepsilon_{uv} & \varepsilon_{vv} \end{bmatrix} = R^{-1}(\theta) \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\varepsilon_{xy} \\ \frac{1}{2}\varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix} R(\theta) \tag{6}$$

Given a set of external forces acting along the edges of a discretized membrane, the equilibrium problem is to find a deformation that induces strains and stresses that balance the external forces. For equilibrium to exist, the external forces and torques must sum to zero.

In the triangulated finite element formulation, the deformations at vertices, $\delta_j, j = 1, \ldots, n$, are related to the external forces loaded on those vertices, $f_i, i = 1, \ldots, n$, through a *stiffness matrix* $\mathbf{K}$ with elements K_{ij}. However, the elements K_{ij} are not scalars. Since δ_j and f_i are 2-vectors, K_{ij} is best thought of as a *linear transformation* from E^2 to E^2. This view is independent of the coordinate frame. If a coordinate frame is chosen, then of course K_{ij} can be *represented by* a 2×2 matrix in that frame. The interpretation of the linear transformation K_{ij} is: $K_{ij}\delta_j$ is the elastic force experienced at vertex i due to a displacement δ_j of vertex j. Letting δ and f be the vectors of δ_j and f_i, in equilibrium, we have

$$\mathbf{K}\delta + f = 0 \tag{7}$$

The form of K_{ij} depends on the finite element model being used.

5 Exact Simulation by Springs is Impossible

This section demonstrates the difficulties encountered in an attempt to simulate a linearly elastic material accurately by a spring mesh, using the same triangulated surface for both models. The problems are shown to arise even in the simplest case: a two-dimensional planar membrane of constant thickness, with the simplest membrane model (constant strain), and with uniform elasticity in the membrane.

The finite element model most often seen for membranes is commonly called the *constant strain* model [BR75, Whi85, Fen86, ZT89]. Each triangle's stress and strain functions are assumed to be constant over the triangle's surface. This ensures that each triangle deforms into a triangle, and the global deformation is a piecewise continuous function of position. Each triangle edge has a uniform force per unit length, which is the sum of an internal forces due to the stress and "body force" in the triangle, and external forces applied to that edge. "Body force" is usually weight. In equilibrium, the internal and external force at the edge sum to zero. The **K** matrix is computed by computing the contribution of each triangle separately, and summing.

Let the triangle T_e have vertices p, q, and r in counter-clockwise order. Fix a coordinate frame (x, y) for definiteness. We use the notations (x_p, y_p) for the position of p and (x_{qp}, y_{qp}) as an abbreviation for $(x_q - x_p, y_q - y_p)$. Then $\mathbf{K}_e^{memb}$ is a 3×3 matrix, whose elements are representable in the (x, y) frame as 2×2 matrices.

$$\mathbf{K}_e^{memb} = \begin{bmatrix} K_{pp}^{memb} & K_{pq}^{memb} & K_{pr}^{memb} \\ K_{qp}^{memb} & K_{qq}^{memb} & K_{qr}^{memb} \\ K_{rp}^{memb} & K_{rq}^{memb} & K_{rr}^{memb} \end{bmatrix} \tag{8}$$

where

$$K_{pp}^{memb} = \frac{E_2}{4\Delta(1-\nu^2)} \begin{bmatrix} y_{rq}^2 + \left(\frac{1-\nu}{2}\right) x_{rq}^2 & -\left(\frac{1+\nu}{2}\right) y_{rq} x_{rq} \\ -\left(\frac{1+\nu}{2}\right) y_{rq} x_{rq} & x_{rq}^2 + \left(\frac{1-\nu}{2}\right) y_{rq}^2 \end{bmatrix} \tag{9}$$

$$K_{pq}^{memb} = \frac{E_2}{4\Delta(1-\nu^2)} \begin{bmatrix} y_{rq} y_{pr} + \left(\frac{1-\nu}{2}\right) x_{rq} x_{pr} & -\nu y_{rq} x_{pr} - \left(\frac{1-\nu}{2}\right) y_{pr} x_{rq} \\ -\nu x_{rq} y_{pr} - \left(\frac{1-\nu}{2}\right) x_{pr} y_{rq} & x_{rq} x_{pr} + \left(\frac{1-\nu}{2}\right) y_{rq} y_{pr} \end{bmatrix} \tag{10}$$

and other components are found by cyclic permutation of (p, q, r). In the above equations, E_2 is the two-dimensional Young's modulus, and ν is the Poisson coefficient, both being properties of the material. Also, Δ denotes the area of the triangle, following finite-element notation.

Now let us consider a different elasticity problem, analysis of an arrangement of pin-jointed trusses. This problem is equivalent what is usually called a "spring mesh" in computer graphics literature. That is, each elastic element is one-dimensional, elements are "pinned" together at the vertices, so they may rotate freely. Stretching or compressing an element from its "rest length" produces a one-dimensional stress. We shall begin with a planar arrangement, with linear elasticity. This is one of the earliest problems to be analyzed with the finite element method [TCMT56].

Let us consider the same triangle as above: edges (p, q), (q, r), and (r, p), in their rest positions. In addition, we specify the rest lengths and spring stiffness coefficients as L_{qp}, k_{qp}, etc. We continue to abbreviate $(x_q - x_p)$ as x_{qp}, etc. The stiffness matrix for the triangle of springs (or trusses), $\mathbf{K}_e^{spring}$, is again a 3×3 matrix whose elements are represented in the (x, y) frame as 2×2 matrices. We have [BR75, ZT89]:

$$K_{pp}^{spring} = E_1 \begin{bmatrix} \dfrac{k_{qp} x_{qp}^2}{L_{qp}^3} + \dfrac{k_{pr} x_{pr}^2}{L_{pr}^3} & \dfrac{k_{qp} x_{qp} y_{qp}}{L_{qp}^3} + \dfrac{k_{pr} x_{pr} y_{pr}}{L_{pr}^3} \\ \dfrac{k_{qp} x_{qp} y_{qp}}{L_{qp}^3} + \dfrac{k_{pr} x_{pr} y_{pr}}{L_{pr}^3} & \dfrac{k_{qp} y_{qp}^2}{L_{qp}^3} + \dfrac{k_{pr} y_{pr}^2}{L_{pr}^3} \end{bmatrix} \tag{11}$$

$$K_{pq}^{spring} = E_1 \begin{bmatrix} -\dfrac{k_{qp} x_{qp}^2}{L_{qp}^3} & -\dfrac{k_{qp} x_{qp} y_{qp}}{L_{qp}^3} \\ -\dfrac{k_{qp} x_{qp} y_{qp}}{L_{qp}^3} & -\dfrac{k_{qp} y_{qp}^2}{L_{qp}^3} \end{bmatrix} \tag{12}$$

and other components are found by cyclic permutation of (p, q, r).

Now, if we want the spring triangle to simulate the constant strain triangle, the obvious approach is to try to choose k_{qp}, k_{rq}, k_{pr}, and E_1 to bring the stiffness matrices of the two models into agreement. However, by inspecting the two constraints

$$K_{pq,11}^{spring} = K_{pq,11}^{memb} \qquad K_{pq,12}^{spring} = K_{pq,12}^{memb}$$

we see that, in general, no such choice is possible. In particular, if $p = (0, 0)$, $q = (1, 0)$ and $r = (0, 1)$, then the second constraint is not satisfied by any choice of spring constants.

In summary, examination of the stiffness matrices associated with the constant strain model and the spring-mesh model shows that, in general, no assignments of individual stiffness coefficients enable the stiffness matrix of the latter to agree with that of the former.

However, there is a possible saving clause. The stiffness matrices actually are singular, and therefore are not unique representations of the constraints. Therefore, it is still possible that systems with differing stiffness matrices may have the same equilibria. We now show that this is not the case.

Example 5.1: Consider again an isosceles right triangle, only this time, for convenience, with $p = (0, 0)$, $q = (1, 1)$ and $r = (-1, 1)$. Considering this as a membrane, assume a uniform, tensile, boundary stress of magnitude $\sqrt{2}$ is applied horizontally on edges (p, q) and (r, p), as shown in Figure 2. This is equivalent to horizontal tensile forces of magnitude 1 being applied at vertices q and r in the spring-mesh model. The two opposing forces at p cancel.

For the membrane model, the internal stress at equilibrium is $\sigma_{xx} = 2$, $\sigma_{yy} = 0$, and $\sigma_{xy} = 0$. Therefore, the *strain* is $\varepsilon_{xx} = 2/E_2$, $\varepsilon_{yy} = -2\nu/E_2$, and $\varepsilon_{xy} = 0$. In other words, if edge (q, r) increases in length by some small amount, $2\varepsilon_{xx}$, then the height (in y) decreases by $\nu\varepsilon_{xx}$.

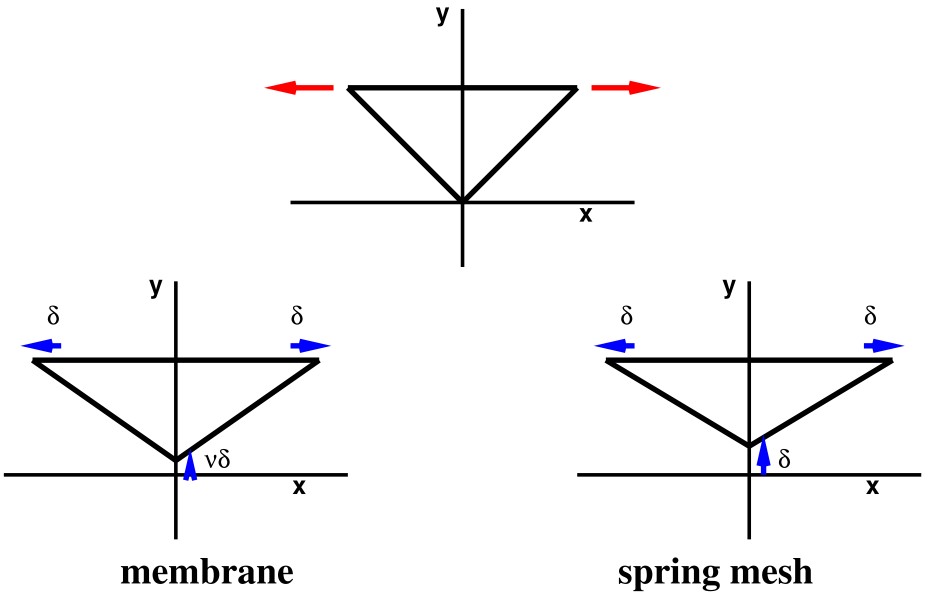

Figure 2: Isosceles right triangle subjected to horizontal forces (above) deforms differently under the membrane model (left) and the spring-mesh model (right). See Section 5 for discussion.

However, for the spring-mesh model, since there is no force on p, the edges (p,q) and (r,p) are under no tension and do not change length, despite the horizontal forces at q and r. It follows that an increase in length of (q,r) by any small amount, 2δ, is accompanied by a decrease in height of δ. But $\nu \leq \frac{1}{2}$ for all physical materials. Therefore, no matter how the spring stiffness for (q,r) is chosen, the spring-mesh deformation cannot agree with the membrane deformation for horizontal external forces. $\square$

6 Constant Strain Approximation by Spring Meshes

Let us consider from first principles the problem of assigning spring stiffness coefficients to the edges of a triangle in such a way that the triangle will deform in the same manner as an isotropic elastic membrane, at least to the first order effects, as modeled by the constraint strain model. That is, we limit consideration to the case that the external forces are uniformly distributed along each edge (different edges may have different forces – only the distribution is uniform). Also, we are concerned only with linear, equilibrium deformations. Therefore, we assume that the net external force and torque are zero.

Consider a triangle with edges a, b, and c in counter-clockwise order. Let α, β, and γ be the opposing angles, as shown in Figure 3, left. Now suppose opposing external forces are applied at the ends of edge c, say a stretching force, for definiteness.

First, suppose that the triangle consists of pin-connected springs (or trusses). Then, clearly edge c will elongate, and edges a and b will not change length. Moreover, edge c will remain horizontal. Assuming a

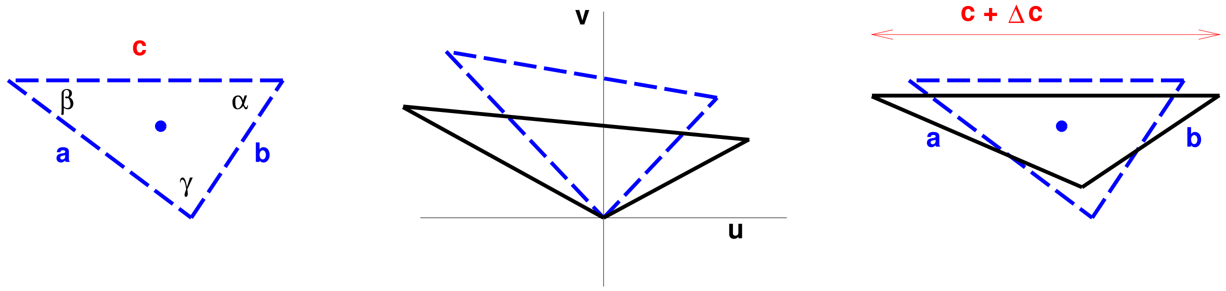

Figure 3: Undeformed triangular membrane (left, dashes), is rotated into a frame such that the v-axis bisects the angle γ (center). A constant strain with u and v axes as principal directions deforms the triangle (center, solid figure), which is then rotated back to make the upper edge horizontal (right, solid figure). The second rotation is slightly different from the inverse of the first. The overall effect is the same as if the triangle were a spring mesh and a stretching force were applied horizontally at the ends of edge c. See Section 6 for discussion.

uniform distribution of mass, the center of gravity will not move, since no net work is done on the body. Therefore the black triangle in the right part of the figure represents the equilibrium position. Notice that the lower vertex elevates by exactly twice the amount that edge c descends, to maintain the center of gravity. The amount of elongation of edge c is defined as $\Delta c/c$, where we use "c" both to denote the edge and its length. This elongation depends only on c and k_c, the spring stiffness for edge c.

Now, suppose that the triangle consists of a membrane modeled by the constant strain model. The horizontal opposing external forces are distributed uniformly along edges a and b with force-per-unit-length of $-\mathbf{f}/a$ and $\mathbf{f}/b$, where $\mathbf{f}$ is a force in the positive horizontal direction. These boundary forces must be balanced by internal stress to achieve equilibrium.

The question is, for what family of internal stresses will the corresponding strains be such that the lengths of a and b do not change? This question has an elegant geometrical answer, which is indicated in the center panel of Figure 3. The key is to rotate into the coordinate frame (u, v) such that the v-axis bisects the angle γ (after that vertex is translated to the origin).

Now, *any* constant stress whose principal directions are the u and v axes will produce identical elongations in edges a and b. (Recall that elongations are $\Delta a/a$ and $\Delta b/b$, not Δa and Δb.) This follows from the observation that the slopes of edges a and b are equal in magnitude, although opposite in sign. For example, a positive stress σ_{uu} increases the u-location of the rightmost vertex by $b \sin(\gamma/2)\sigma_{uu}/E_2$ and decreases the u-location of the leftmost vertex by $a \sin(\gamma/2)\sigma_{uu}/E_2$. This stress *decreases* the v-locations of the rightmost and leftmost vertices by $\nu b \cos(\gamma/2)\sigma_{uu}/E_2$ and $\nu a \cos(\gamma/2)\sigma_{uu}/E_2$, respectively. By similar triangles, the *relative* change in length, which is the strain, is the same for edges a and b. Similar remarks apply to a stress σ_{vv}. For small elongations, the effects of σ_{uu} and σ_{vv} may be superposed.

Since any constant stress whose principal directions are the u and v axes will produce identical elongations in edges a and b, it suffices to characterize the family of such stresses that produce *zero* elongation in edge b. For small elongations in the u and v directions, the elongation of edge b is zero just when the rightmost vertex is displaced in a direction orthogonal to edge b; i.e.,

$$\frac{\Delta v}{\Delta u} = \frac{-\sin(\gamma/2)}{\cos(\gamma/2)} \tag{13}$$

The family of stresses that satisfies this condition is given by

$$\frac{\cos(\gamma/2)(-\nu\sigma_{uu}+\sigma_{vv})}{\sin(\gamma/2)(\sigma_{uu}-\nu\sigma_{vv})} = \frac{-\sin(\gamma/2)}{\cos(\gamma/2)} \tag{14}$$

which can be rewritten as

$$\sigma_{vv}((1+\nu)\cos\gamma + (1-\nu)) = \sigma_{uu}((1+\nu)\cos\gamma - (1-\nu)) \tag{15}$$

Letting $(\Delta u, \Delta v)$ denote the displacement of the rightmost vertex, and letting Δc denote the change in the *length* of edge c, for small elongations (ignoring second order terms), we have

$$\Delta c = 2\sin(\alpha)\Delta u\sqrt{1+\left(\frac{\Delta v}{\Delta u}\right)^2} = \frac{2\sin(\alpha)\Delta u}{\cos(\gamma/2)} \tag{16}$$

where the relation $(\sin\alpha/a) = (\sin\beta/b)$ was used to eliminate $\sin\beta$. But

$$\Delta u = \left(\frac{b\sin(\gamma/2)}{E_2}\right)(\sigma_{uu}-\nu\sigma_{vv}) \tag{17}$$

Now, let f denote the magnitude of the force stretching edge c (with $f<0$ to denote a compression force). In the spring mesh model, let k_c be the spring stiffness coefficient of edge c. Then

$$f = k_c\Delta c \tag{18}$$

Now, switching to the constant strain membrane model, define $\theta = (\alpha-\beta)/2$. This is the direction parallel to edge c, relative to the u-axis. Then the stress in direction θ, denoted as σ_θ, is given by Eq. 5

$$\sigma_\theta = \cos^2\theta\sigma_{uu} + \sin^2\theta\sigma_{vv} \tag{19}$$

Next, σ_{vv} can be eliminated from the above equation by use of Eq. 15, giving:

$$\sigma_\theta = \left(\frac{(1+\nu)\cos\gamma + (1-\nu)(\cos(\alpha-\beta))}{(1+\nu)\cos\gamma+(1-\nu)}\right)\sigma_{uu} \tag{20}$$

Now, the boundary condition requires [TG51]

$$\sigma_\theta = \frac{f}{a\sin\beta} = \frac{f}{b\sin\alpha} \tag{21}$$

Combining Eqs. 15–21, we obtain

$$\frac{f}{k_c} = \frac{2b\sin\alpha\sin(\gamma/2)}{E_2\cos(\gamma/2)}\left(1-\nu\frac{\sigma_{vv}}{\sigma_{uu}}\right)\sigma_{uu} \tag{22}$$

$$\frac{1}{k_c} = \frac{2\sin(\gamma/2)}{E_2\cos(\gamma/2)}\left(\frac{(1-\nu^2)(1+\cos\gamma)}{(1+\nu)\cos\gamma+(1-\nu)\cos(\alpha-\beta)}\right) \tag{23}$$

Solving for k_c:

$$k_c = \frac{E_2}{(1-\nu^2)}\left(\frac{\sin\alpha\sin\beta - \nu\cos\alpha\cos\beta}{\sin\gamma}\right) \tag{24}$$

Finally, using standard geometric identities:

$$k_c = \left(\frac{E_2}{1+\nu}\right)\frac{2\,area(T_e)}{c^2} + \left(\frac{E_2\,\nu}{1-\nu^2}\right)\frac{(a^2+b^2-c^2)}{4\,area(T_e)} \tag{25}$$

Corresponding formulas may be obtained for k_a and k_b by the obvious renaming.

The coefficient derived, k_c applies for *one* triangle of which c is an edge, but unless c is a boundary edge, c occurs in *two* triangles, so the two individual coefficients are added to give the total stiffness coefficient for edge c in the overall spring mesh.

7 Discussion and Conclusions

The method described has been implemented for some test meshes, where the correct behavior is known from the construction. For simplicity we used $\nu = 0$ throughout the tests. Test results are illustrated in Figs. 4 and 5.

In the first test a simple, hexagonal planar membrane was simulated. As shown in Figure 4, the discretization was irregular. The external forces were equal radial forces, applied at the six boundary vertices. Assuming a uniformly elastic membrane, the deformation should also be uniform, at least with six rotational symmetries. When all edges were assigned the same stiffness, noticeable distortion occurs at equilibrium. This distortion exists for smaller deformations as well; it is not merely a manifestation of nonlinearity. When edges were assigned coefficients in accordance with Eq. 25, the distortion disappears.

In the second test, an isosurface was extracted, based on a potential field from two nearby spheres, forming a “skin” around the pair. The isosurface is represented as a triangulated mesh. Each triangle has a circle textured upon it, in its undeformed shape (some triangles did not get an entire circle). The widely varying sizes of the circles indicate the widely varying sizes of triangles in the isosurface. Because of the popularity of isosurfaces as a means of surface creation, we believe this test is indicative of effects that can be expected in practice. To induce a deformation, one sphere was pulled away from the other, with the “skin” constrained to stretch so that it surrounded them both. Then the skin was allowed to reach equilibrium, based on the spring mesh model. Now, for a uniformly elastic membrane the skin should deform fairly uniformly. At least, the deformations of nearby triangles should be approximately the same. The deformed circles become ellipses that demonstrate the effective strain in each triangle. Again, we see that setting all spring coefficients equal leads to widely varying strains. However, the use of Eq. 25 to assign coefficients produced significantly more uniform strains in the individual triangles.

In conclusion, this paper has derived an approximate formula for defining spring stiffness coefficients for each edge in terms of the geometry of the mesh triangles containing that edge. The resulting spring mesh approximates an isotropic elastic membrane, at least for small deformations. Experimental results presented indicate that the agreement is quite good for fairly large deformations as well.

Acknowledgments

This research was funded in part by Research and Development Laboratories, and by NSF Grant CDA-9115268.

References

[BR75] William H. Bowes and Leslie T. Russel. *Stress Analysis by the Finite Element Method for Practicing Engineers.* D. C. Heath, Lexington, 1975.

[CG91] G. Celniker and G. Gossard. Deformable curve and surface finite elements for free-form shape design. *Computer Graphics (ACM SIGGRAPH Proceedings)*, 25:257–266, July, 1991.

[CZ92] David T. Chen and David Zeltzer. Pump it up: Computer animation based model of muscle using the finite element method. *Computer Graphics (ACM SIGGRAPH Proceedings)*, 26(2):89–98, July 1992.

[Fen86] R. T. Fenner. *Engineering Elasticity : Application of Numerical and Analytical Techniques.* Ellis Horwood Series in Mechanical Engineering. John Wiley, New York, 1986.

[KGC+96] R. M. Koch, M. H. Gross, F. R. Carls, D. F. von Buerin, G. Fankhauser, and Y. I. H. Parish. Simulating facial surgery using finite element models. *Computer Graphics (ACM SIGGRAPH Proceedings)*, pages 421–428, August 1996.

[LK95] Jintae Lee and Tosiyasu Kunii. Model-based analysis of hand posture. *IEEE Computer Graphics and Applications*, 15(5):77–86, 1995.

[LTW95] Yuencheng Lee, Demetri Terzopoulos, and Keith Waters. Realistic modeling for facial animation. *Computer Graphics (ACM SIGGRAPH Proceedings)*, pages 55–62, August 1995.

[Mil88] Gavin Miller. From wire-frames to furry animals. In *Graphics Interface '88*, pages 138–145, Edmonton, Alberta, June 1988.

[TCMT56] M. J. Turner, R. W. Clough, H. C. Martin, and L. C. Topp. Stiffness and deflection analysis of complex structures. *J. Aeronautical Science*, 23(9), September 1956.

[TF88] Demetri Terzopoulos and Kurt Fleischer. Modeling inelastic deformation: Viscoelasticity, plasticity, fracture. *SIGGRAPH '88 Conference Proceedings*, 22(4):269–278, August, 1988.

[TG51] S. Timoshenko and J. N. Goodier. *Theory of Elasticity*. McGraw-Hill Book Company, Inc., New York, second edition edition, 1951.

[TPBF87] Demetri Terzopoulos, John Platt, Alan H. Barr, and Kurt Fleischer. Elastically deformable models. *SIGGRAPH '87 Conference Proceedings*, 21(4):214, July 1987.

[TTG95] Demetri Terzopoulos, Xiaoyuan Tu, and Radek Grzeszczuk. Artificial fishes: Autonomous locomotion, perception, behavior, and learning in a simulated world. *Artificial Life*, 1:327–351, 1995.

[TW88] Demetri Terzopoulos and Andrew Witkin. Physically based models with rigid and deformable components. *IEEE Computer Graphics and Applications*, 8(6):41–51, November 1988.

[TW90] D. Terzopoulos and K. Waters. Physically-based facial modelling, analysis, and animation. *The Journal of Visualization and Computer Animation*, 1(2):73–80, 1990.

[TW91] Demetri Terzopoulos and Keith Waters. Techniques for realistic facial modeling and animation. In Nadia Magnenat-Thalmann and Daniel Thalmann, editors, *Computer Animation '91*, pages 58–74, Geneva, 1991. Springer-Verlag.

[VCMT95] Pascal Volino, Martin Courchesne, and Nadia Magnenat-Thalmann. Versatile and efficient techniques for simulating cloth and other deformable objects. *Computer Graphics (ACM SIGGRAPH Proceedings)*, pages 137–144, August 1995.

[Whi85] R. E. White. *An Introduction to the Finite Element Method with Applications to Nonlinear Problems*. John Wiley, New York, 1985.

[ZT89] O. C. Zienkiewicz and R. L. Taylor. *The Finite Element Method*. McGraw-Hill, New York, 4th edition edition, 1989.

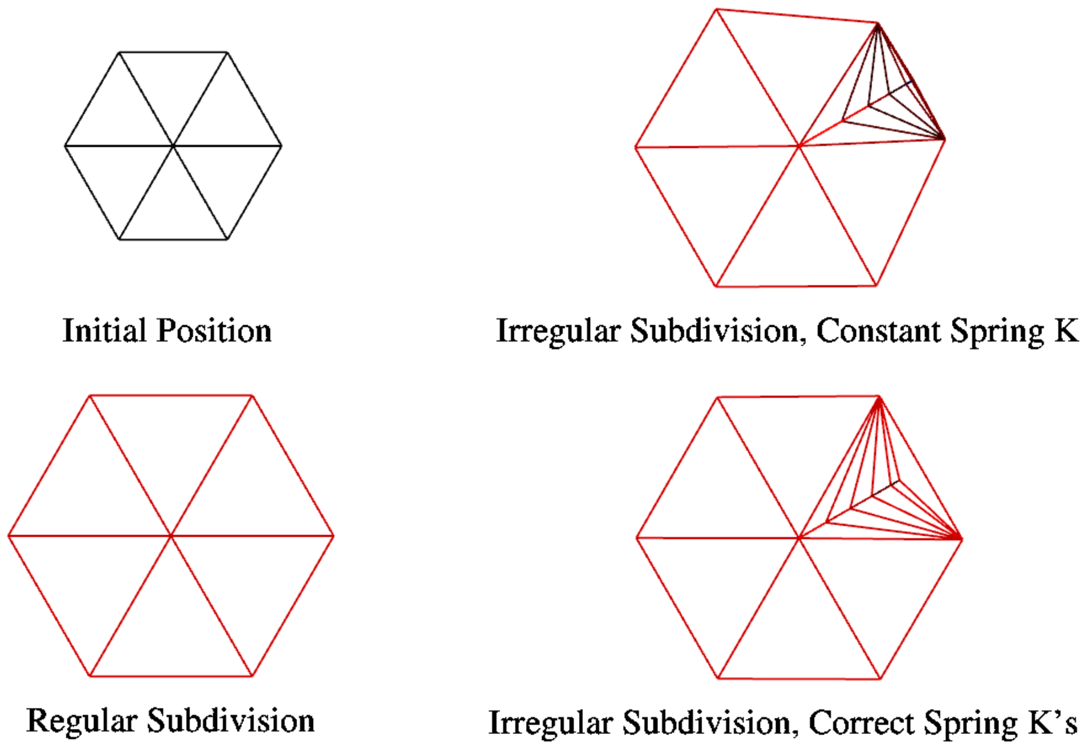

Figure 4: Comparison of effect of spring k's on test membrane. The hexagonal membrane at upper left, which is assumed to be of a homogeneous material, is subjected to a constant outward radial force at the six external vertices. At lower, the regularly triangulated membrane expands correctly to a larger hexagon. At upper right, the irregularly subdivided membrane expands non-uniformly when spring k's are constant for all edges. At lower left, despite being irregularly subdivided, the membrane expands correctly when the spring k's are calculated correctly, as discussed in the body of the paper.

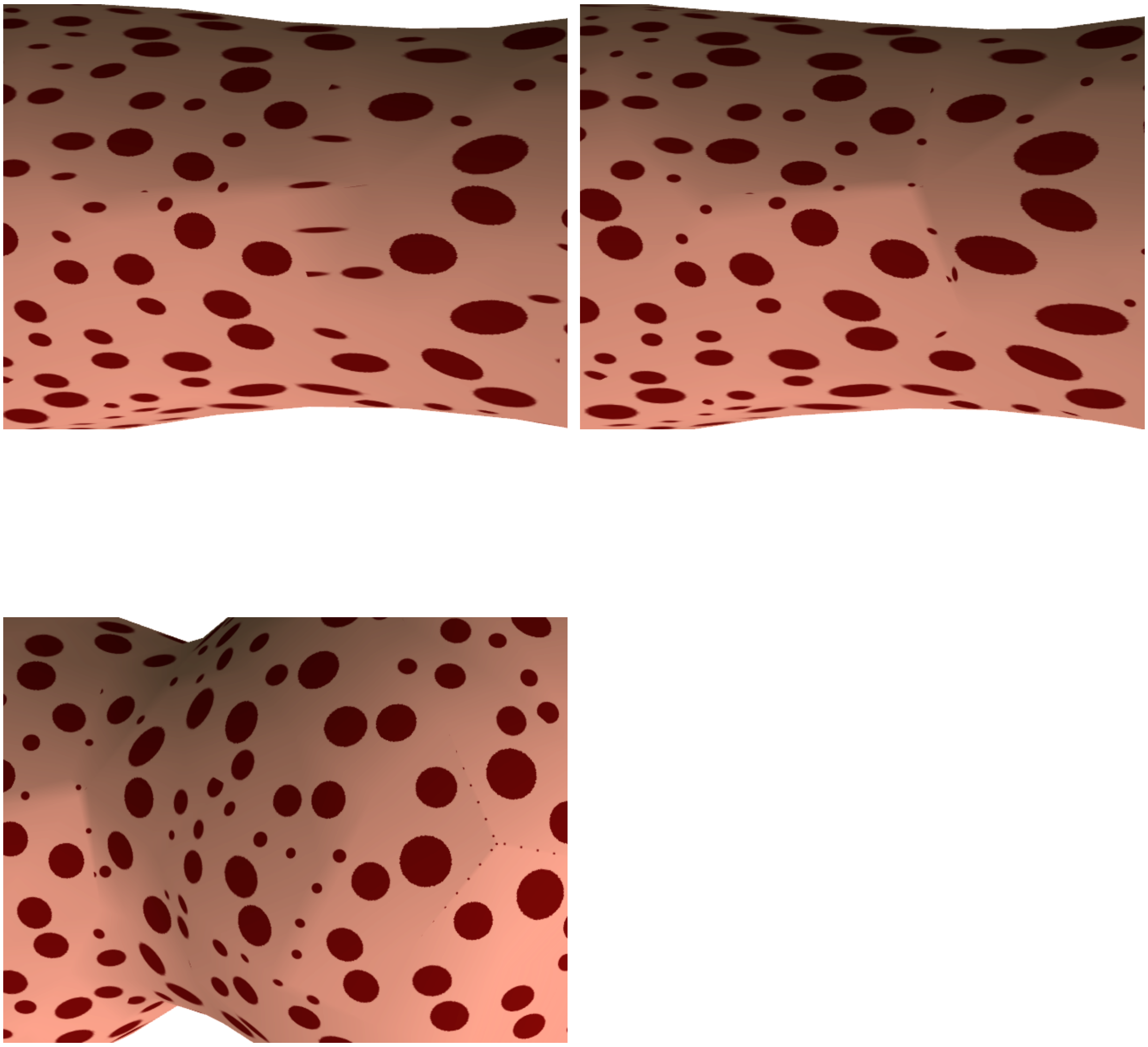

Figure 5: This shows the effect of the spring k's on a stretched elastic membrane. In the initial triangulated mesh, each triangle contained a circle. Because the mesh is quite irregular, the size of the triangles varies. The initial configuration is shown at lower left. The stretched configuration at equilibrium is shown at the top. The stretched configuration using a constant spring k is shown at top left; notice many of the small circles are distorted irregularly. In the top right, correct values are used for spring k's, and the circles are distorted more regularly.