Constructive Mathematics

A Survey

Constructive Proofs - An Example

Problem: Does there exist an irrational number that we can raise to irrational power and get a rational number?

- Classical approach uses excluded middle
- Constructive proof requires a little more work

Constructive Proofs - Methods

- Law of excluded middle (LEM) forbidden
  - Disjunctions (P or not P may not be true)
  - Conditionals and proof by contradiction (If not P implies a contradiction, we haven’t proven P)
- Existential quantifiers and “witnesses”
  - In order to prove that an object exists we need to construct it

What is Constructivism?

- An approach to mathematics that admits only constructive proofs
- A movement which seeks to restructure mathematics along constructive lines
  - First arose as a movement in reaction to Hilbert’s Formalism
  - Proponents claim to keep math practical and intuitive

Constructivism - Obstacles

More nuanced definitions become necessary

- Avoid negative statements (Ex: inequality)
- Ex: subfinite vs finite

We have to become comfortable with ambiguity

- Ex: it's possible for a number to be not positive, not negative, and not equal zero

Obstacle: The Axiom of Choice

For any collection of nonempty sets, X, there exists a choice function that maps every element, s, of X to an element of s.

We will not use AC in our constructions.

- Controversial
- Implies the law of the excluded middle

Problems in Analysis

- Axiom - Every bounded set of real numbers has a supremum and an infimum
  - Constructively invalid
  - Replace with the concept of totally bounded

A set of real numbers, A, is said to be totally bounded if for each $\varepsilon > 0$ there exist a subfinite set of points $\{x_1, \ldots, x_n\}$ in A such that for every $x \in A$, at least one of the values $|x_1 - x|, \ldots, |x_n - x|$ is less than ε. A set that is totally bounded has a supremum and infimum.

Problems in Analysis

-Constructive definition of continuous functions:

A real-valued function, f, is continuous on a closed interval, I, if for each $\varepsilon > 0$ there exists $\omega(\varepsilon) > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| \leq \omega(\varepsilon)$. The function $\omega(\varepsilon)$ is called a **modulus of continuity** for f.

-A theorem which does not exist in classical analysis:

If $f : [a, b] \rightarrow R$ is continuous, then the quantities $sup\, f = sup\{f(x) : x \in [a, b]\}$ and $inf\, f = inf\{f(x) : x \in [a, b]\}$ both exist.

Problems in Analysis

-Intermediate Value Theorem

<u>Classical version:</u> If f is a continuous real-valued function on a closed interval $[a, b]$, and $y \in [min\{f(a), f(b)\}, max\{f(a), f(b)\}]$, then there exists $x \in [a, b]$ such that $f(x) = y$.

<u>Constructive version:</u> If f is a continuous real-valued function on a closed interval $[a, b]$, and $y \in [min\{f(a), f(b)\}, max\{f(a), f(b)\}]$, then for each $\varepsilon > 0$ there exists $x \in [a, b]$ such that $|f(x) - y| < \varepsilon$.

Problems in Linear Algebra

Theorem: every vector space has a basis

- Depends on the Axiom of Choice
- It’s possible to have a constructive system where this is still true by redefining what a vector space is.

Other Problems With the Axiom of Choice

- Algebra:
  - Every field has a prime ideal
  - The existence of algebraic closures
  - Group structure of an arbitrary set
- Set Theory:
  - pretty much everything dealing with objects bigger than aleph zero
- Topology