Constructive Proofs - An Example

Problem: Does there exist an irrational number that we can raise to irrational power and get a rational number?

- Classical approach uses excluded middle
- Constructive proof requires a little more work
Constructive Proofs - Methods

- Law of excluded middle (LEM) forbidden
  - Disjunctions (P or not P may not be true)
  - Conditionals and proof by contradiction (If not P implies a contradiction, we haven’t proven P)

- Existential quantifiers and “witnesses”
  - In order to prove that an object exists we need to construct it
What is Constructivism?

- An approach to mathematics that admits only constructive proofs
- A movement which seeks to restructure mathematics along constructive lines
  - First arose as a movement in reaction to Hilbert’s Formalism
  - Proponents claim to keep math practical and intuitive
Constructivism - Obstacles

More nuanced definitions become necessary

○ Avoid negative statements (Ex: inequality)
○ Ex: subfinite vs finite

We have to become comfortable with ambiguity

○ Ex: it’s possible for a number to be not positive, not negative, and not equal zero
Obstacle: The Axiom of Choice

For any collection of nonempty sets, $X$, there exists a choice function that maps every element, $s$, of $X$ to an element of $s$. We will not use AC in our constructions.

- Controversial
- Implies the law of the excluded middle
Problems in Analysis

- Axiom - Every bounded set of real numbers has a supremum and an infimum
  - Constructively invalid
  - Replace with the concept of totally bounded
Problems in Analysis

- Constructive definition of continuous functions:

A real-valued function, $f$, is continuous on a closed interval, $I$, if for each $\varepsilon > 0$ there exists $\omega(\varepsilon) > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| \leq \omega(\varepsilon)$. The function $\omega(\varepsilon)$ is called a modulus of continuity for $f$.

- A theorem which does not exist in classical analysis:

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then the quantities $\sup f = \sup \{f(x) : x \in [a, b]\}$ and $\inf f = \inf \{f(x) : x \in [a, b]\}$ both exist.
Intermediate Value Theorem

Classical version: If $f$ is a continuous real-valued function on a closed interval $[a, b]$, and $y \in [\min\{f(a), f(b)\}, \max\{f(a), f(b)\}]$, then there exists $x \in [a, b]$ such that $f(x) = y$.

Constructive version: If $f$ is a continuous real-valued function on a closed interval $[a, b]$, and $y \in [\min\{f(a), f(b)\}, \max\{f(a), f(b)\}]$, then for each $\varepsilon > 0$ there exists $x \in [a, b]$ such that $|f(x) - y| < \varepsilon$. 

Problems in Analysis

-Intermediate Value Theorem
Theorem: every vector space has a basis

- Depends on the Axiom of Choice
- It’s possible to have a constructive system where this is still true by redefining what a vector space is.
Other Problems With the Axiom of Choice

- **Algebra:**
  - Every field has a prime ideal
  - The existence of algebraic closures
  - Group structure of an arbitrary set

- **Set Theory:**
  - Pretty much everything dealing with objects bigger than aleph zero

- **Topology**