Discrete Ricci Curvature and Ricci Flow for Graph Mining

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Mobile Networks

- Mobile devices roaming around.

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- Vehicles trajectories, human motion behaviors.

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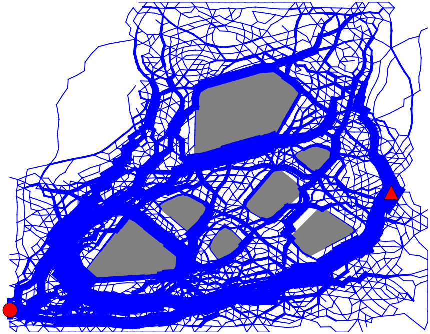

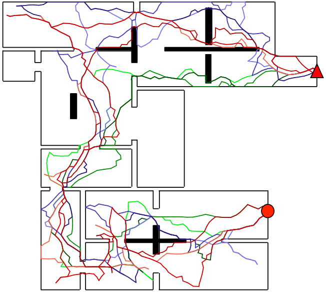

Social Networks

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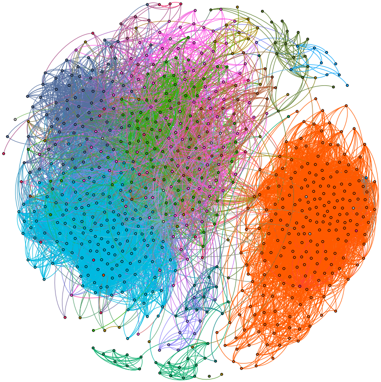
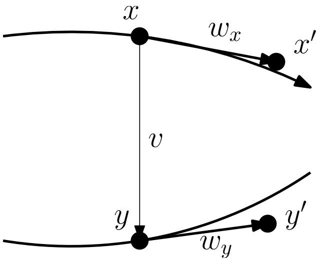


Outline

- Discrete graph curvatures;
- Applications on graph mining and anomaly detection.

Sectional Curvature in Geometry

Consider a tangent vector $v = xy$ and another tangent vector w_x at x. Transport w_x along v to be a tangent vector w_y at y.
If $|x'y'| < |xy|$, then sectional curvature is positive.



Ricci curvature: averaging over all directions w.

Discrete Ricci Curvature

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- Issue: how to match x's neighbors to y's neighbors?
- Assign uniform distribution μ_1, μ_2 on x' and y's neighbors.
- Use optimal transportation distance (earth-mover distance) from μ_1 to μ_2: the matching that minimizes the total transport distance.

Discrete Ricci Curvature

Definition (Ollivier, Lin & Yau)

Let (X, d) be a metric space and let m_1, m_2 be two probability measures on X. For any two distinct points $x, y \in X$, the (Ollivier-) Ricci curvature along xy is defined as

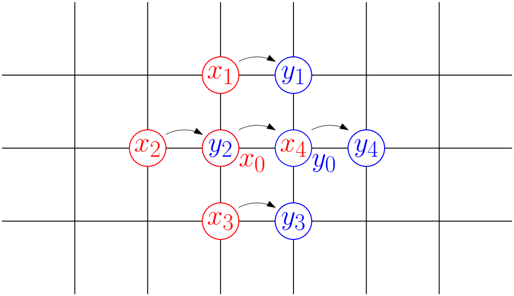
$$\kappa(x, y) := 1 - \frac{W_1(m_x, m_y)}{d(x, y)},$$

where m_x (m_y) is a probability distribution defined on x (y) and its neighbors, $W_1(\mu_1, \mu_2)$ is the L_1 **optimal transportation distance** between two probability measure μ_1 and μ_2 on X:

$$W_1(\mu_1, \mu_2) := \inf_{\psi \in \Pi(\mu_1, \mu_2)} \int_{(u,v)} d(u, v) d\psi(u, v)$$

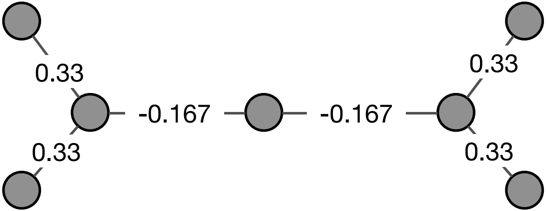
Examples

Zero curvature: 2D grid.



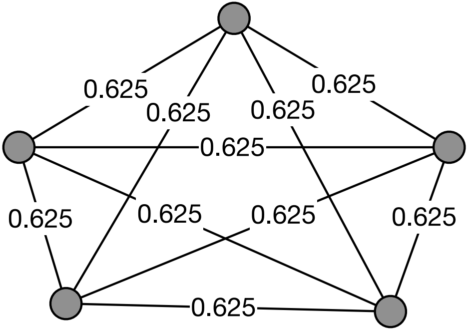
Examples

Negative curvature: tree: $\kappa(x, y) = 1/d_x + 1/d_y - 1$, d_x is degree of x.



Examples

Positive curvature: complete graph.



Data Sets

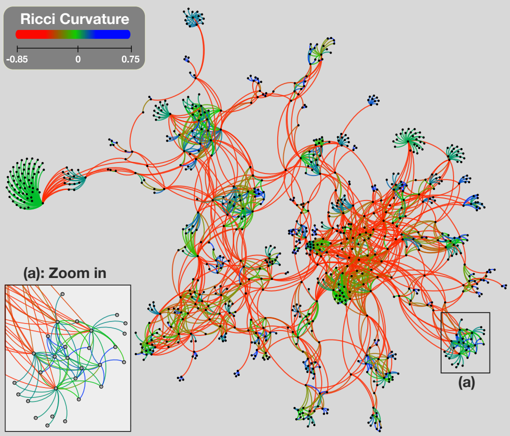
Data Set	Node	Edge	MaxDeg	AvgDeg	Diam	Mean SPL
RouteView	6474	12572	1458	3.88	9	3.71
Gnutella	6301	20777	97	6.59	9	4.64
PGP	10680	24340	205	4.56	24	7.48
Power grid	4941	6594	19	2.67	46	18.99

RocketFuel data set: router level network.

RochetFuel	Node	Edge	MaxDeg	AvgDeg	Diam	Mean SPL
AS: 7018	10145	14166	78	2.04	11	6.95
AS: 3257	843	1156	90	2.23	14	5.27
AS: 3967	895	2071	75	4.63	13	5.94
AS: 1221	2998	3789	106	2.53	12	5.53

Curvature Distribution

Negatively curved edges are like “backbones”, maintaining the connectivity of clusters, in which edges are mostly positively curved.



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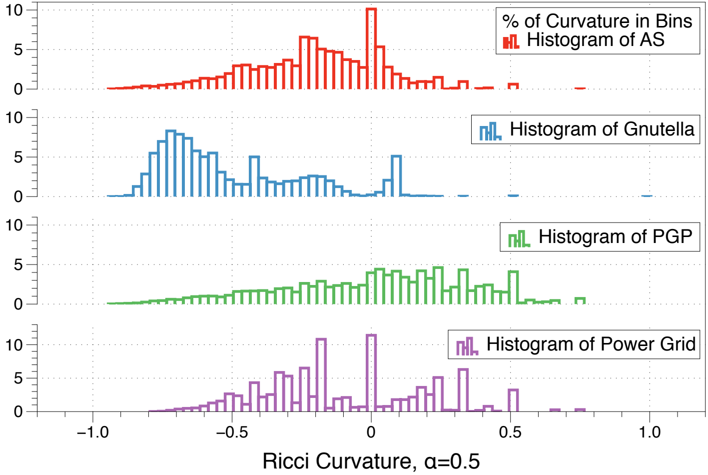
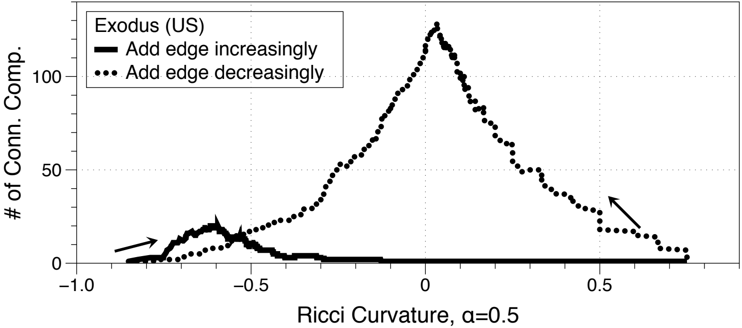


Network Connectivity

Negatively curved edges are well connected. Adding edges with increasing/decreasing curvature: few/many connected components.



Applications: Graph Mining

- Comparing two graphs;
- Graph classification;
- Anomaly detection.

Graph Isomorphism

Given a pair of graphs G_1, G_2, find a one-to-one correspondence of the vertices in G_1 to vertices in G_2 such that (u, v) is an edge in G_1 if and only if their corresponding nodes $f(u), f(v)$ are connected in G_2.

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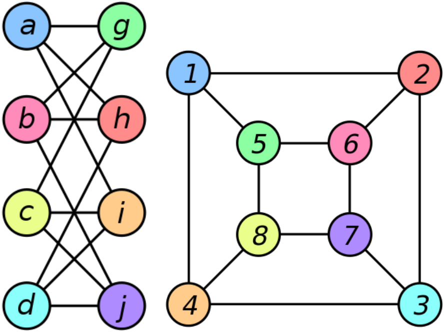

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- Many practical algorithms: e.g., NAUTY.
- Subgraph isomorphism is NP-complete.
- Approximate graph isomorphism: find the best correspondence between vertices in G_1 and G_2.

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- Anomaly?

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Select well positioned nodes as *landmarks* and define the position of a node wrt landmarks.

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Our Idea: Edge Weights Generated by Ricci flow

Given a graph G in which $d(x, y)$ is the weight of the edge xy and $\kappa(x, y)$ is the discrete Ricci curvature, we run
/* Ricci flow

$$d_{i+1}(x, y) = d_i(x, y) - \varepsilon \cdot \kappa_i(x, y) \cdot d_i(x, y)$$

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Ricci Flow Metric: Shortest path metric with edge weights computed above.

[Ni, Lin, G, Gu under submission]

Ricci Flow Metric

$$(d(x,y) - \kappa(x,y) \cdot d(x,y)) \cdot N \approx d(x,y)$$

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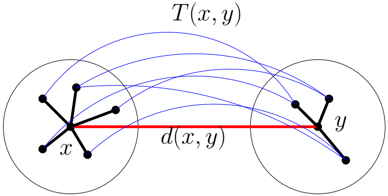
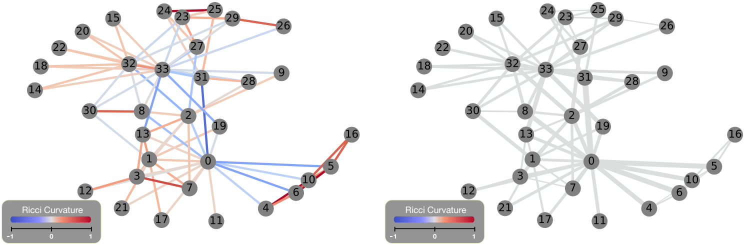
$$\frac{T(x,y)}{d(x,y)} \approx \frac{1}{N}.$$



If (x, y) is removed, there are likely alternative paths from x to y that give similar distance. $\rightarrow$ Much better robustness under edge insertion/deletion.

Ricci Flow Metric



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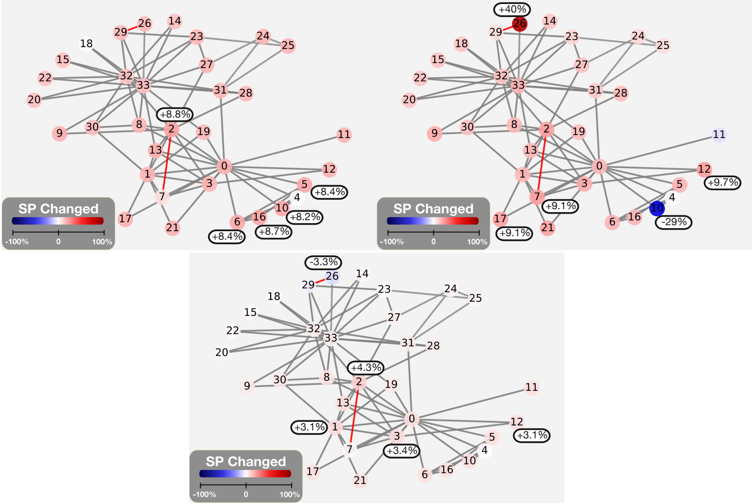

Conclusion

Geometric approaches for network analysis has great potential.

- Community detection.
- Anomaly detection.

Acknowledgement

- Yu-Yao Lin, Dr. Chien-Chun Ni, David Gu (Stony Brook), Feng Luo (Rutgers).

- https://www3.cs.stonybrook.edu/~jgao
- Questions and comments?