Double-diffusive processes in stellar astrophysics

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Lecture 4:
Additional physics

Recap

- Fingering convection well understood:
  - Brown et al. model can be used to estimate mixing by small-scale fingering convection
  - Large-scale structures were shown NOT to form spontaneously at stellar parameters

- Implications:
  - Fingering is quite efficient at draining excess metallicity from surface of stars post-accretion. Implications for planet-bearing stars, WDs, etc…
  - Fingering alone is not sufficient to explain RGB star observations of abundance changes at the luminosity bump.

Recap

- ODDC (semiconvection) is relatively well understood
  - Two regimes: layered and non-layered
  - Mirouh et al. model can be used to determine which regime to expect for given parameters (typically, layered is expected in stellar cores)
  - Wood et al. model for transport by layered convection can be used for mixing. However, layer height remains unknown.

- Implications
  - Mixing is quite efficient, even for very small layer heights
  - Maybe semiconvection can be ignored in intermediate mass stars, and using Schwarzschild criterion is sufficient.

However.....

- Many effects have been ignored !
    - Rotation
    - Shear
    - Magnetic fields
    - Compressibility
    - Chemistry / latent heat effects
    - External perturbations (waves)
    - …

- It is time to look at how they influence double-diffusive instabilities.

Rotation

The effect of (moderate) rotation

In the presence of (moderate) rotation, governing Spiegel-Veronis-Boussinesq equations become

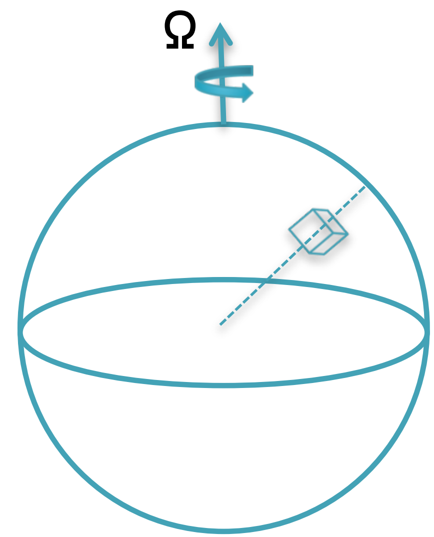
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\Omega \times \mathbf{u} = -\frac{\nabla p}{\rho_m} - \frac{\rho}{\rho_m} g\hat{e}_z + \nu \nabla^2 \mathbf{u}$$

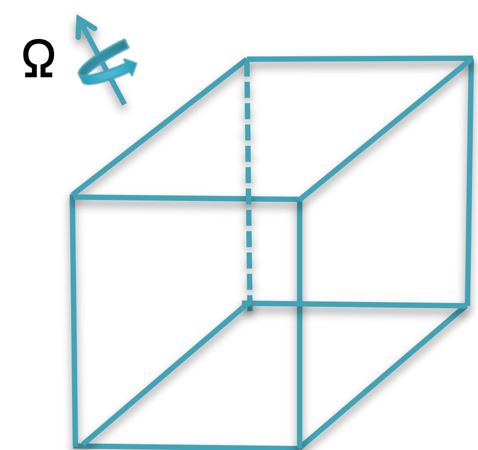
$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T + wT_{0z} = \kappa_T \nabla^2 T$$

$$\frac{\partial C}{\partial t} + \mathbf{u} \cdot \nabla C + wC_{0z} = \kappa_C \nabla^2 C$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\rho}{\rho_m} = -\alpha T + \beta C$$





The Taylor-Proudman constraint (detour)

- In some limit, the effect of rotation dominates over all other ones, and the dominant balance is

$$2\Omega \times \mathbf{u} \cong -\frac{\nabla p}{\rho_m}$$

- Taking the curl to eliminate pressure:

$$\nabla \times \left(2\Omega \times \mathbf{u}\right) \cong 0 \rightarrow \Omega \cdot \nabla \mathbf{u} \cong 0$$

- So the velocity field *must* be invariant in the direction of rotation (Taylor-Proudman constraint).

- This constraint becomes increasingly strong as rotation increases.

The effect of (moderate) rotation

Governing non-dimensional equations *with rotation*

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}+\sqrt{\mathrm{Ta}_*}\,\mathbf{e}_\Omega\times\mathbf{u}\right)=-\nabla p+\left(T-C\right)\mathbf{e}_\mathbf{z}+\nabla^2\mathbf{u}$$

$$\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T\pm w=\nabla^2 T$$

$$\frac{\partial C}{\partial t}+\mathbf{u}\cdot\nabla C\pm\frac{w}{R_0}=\tau\nabla^2 C$$

$$\nabla\cdot\mathbf{u}=0$$

Governing parameters:

$$\mathrm{Pr}=\frac{\nu}{\kappa_T},\quad \tau=\frac{\kappa_C}{\kappa_T}$$

$$R_0=\frac{\alpha\left(T_{0z}-T_z^{ad}\right)}{\beta C_{0z}}$$

$$\mathrm{Ta}_*=\frac{4\Omega^2 d^4}{\kappa_T^2}$$

$$[l]=d=\left(\frac{\kappa_T\nu}{\alpha g\left|T_{0z}-T_z^{ad}\right|}\right)^{1/4},$$

$$[t]=\frac{d^2}{\kappa_T},\quad [T]=d\left|T_{0z}-T_z^{ad}\right|,\quad [C]=\frac{\alpha}{\beta}d\left|T_{0z}-T_z^{ad}\right|$$

The effect of (moderate) rotation

Note that

$$\mathrm{Ta}_* = \frac{4\Omega^2 d^4}{\kappa_T^2} = \frac{4\Omega^2}{N^2}\frac{\nu}{\kappa_T} = 4\cdot 10^{-12}\left(\frac{\Omega}{10^{-6}}\right)^2\left(\frac{N^2}{10^{-6}}\right)^{-1}\left(\frac{\nu}{10}\right)\left(\frac{\kappa_T}{10^7}\right)^{-1}$$

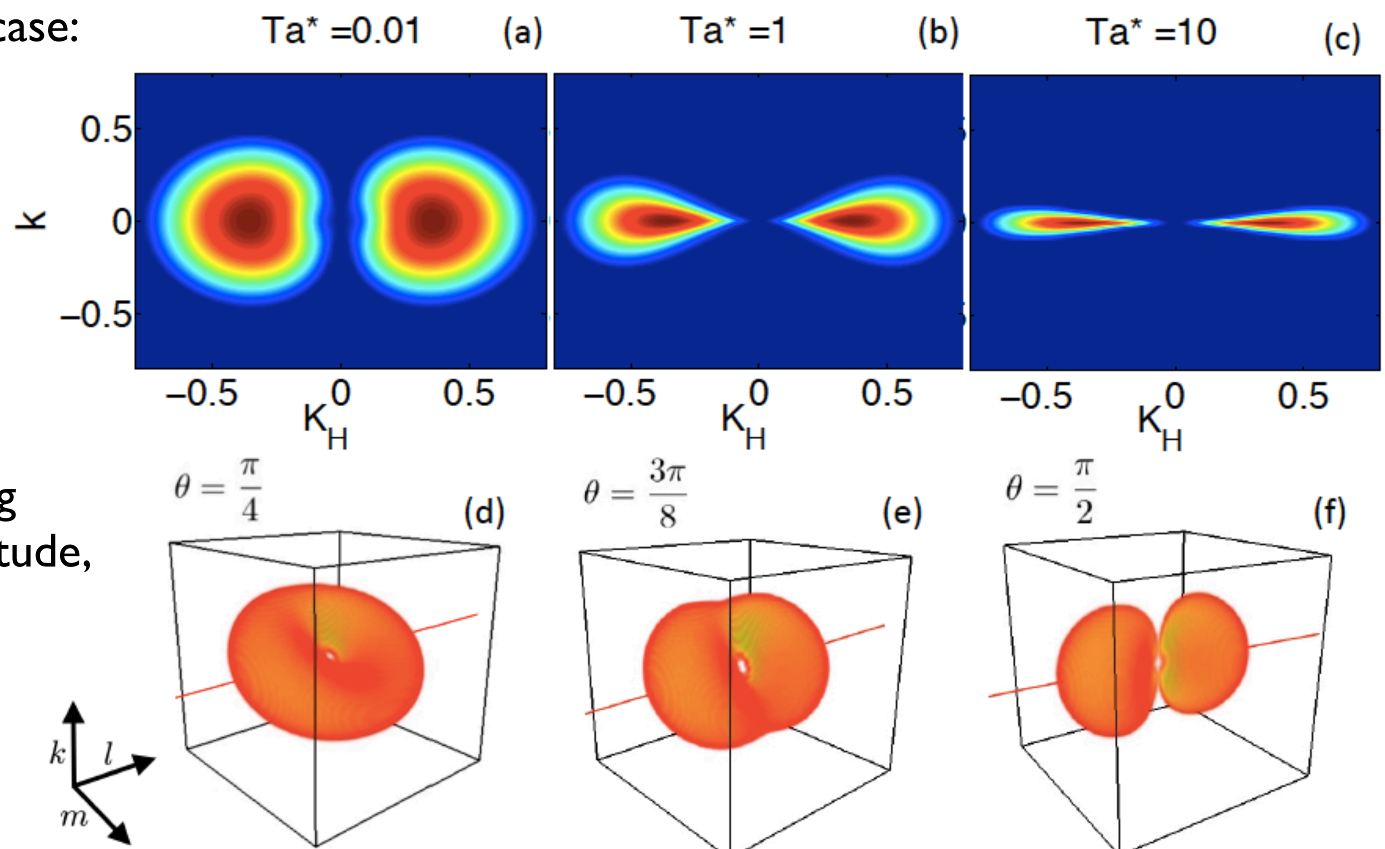
so one may naively think that the effect of rotation is negligible in stars. However this is not true (see later)

The effect of rotation on linear stability

- As expected, rotation suppresses linearly unstable modes that are not invariant along the rotation axis, for both fingering and ODDC

Polar case:

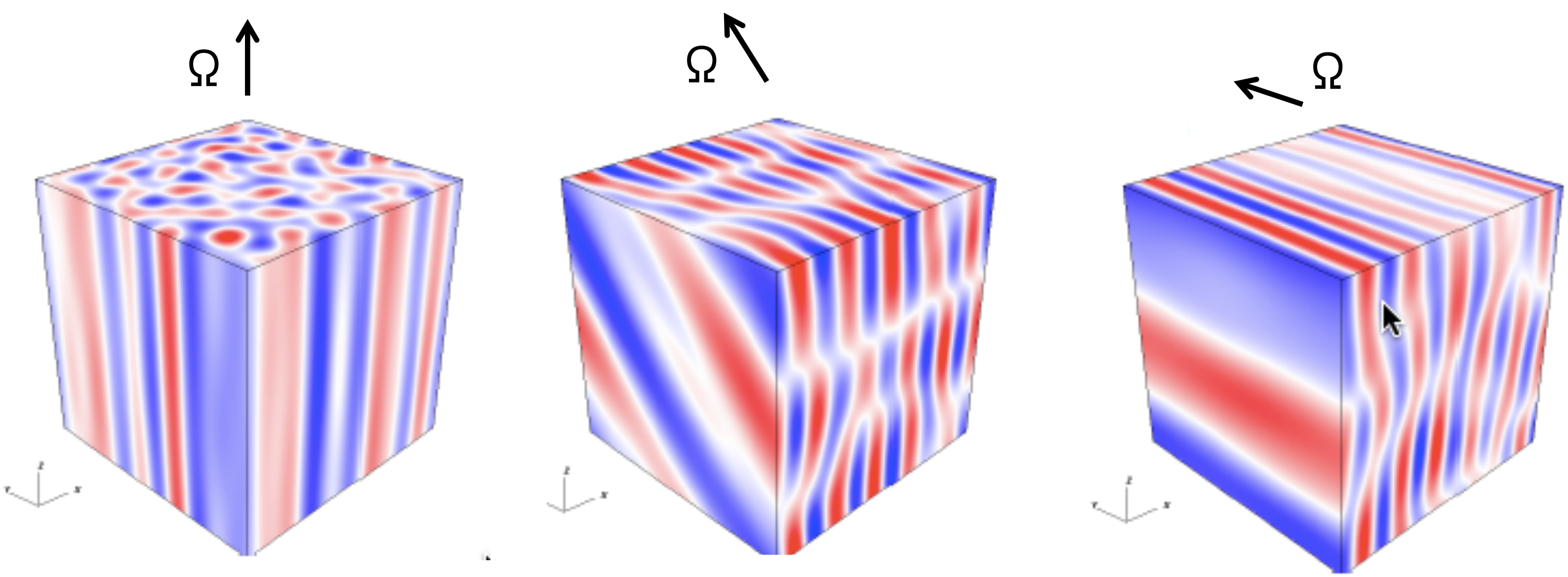
Varying co-latitude, Ta*=1



ODDC case, Moll et al, 2017

The effect of rotation on linear stability

- Fastest-growing modes invariant in the *plane* spanned by Ω and $\mathbf{g}$
- Their growth rates are *unaffected by rotation* (can easily be shown from linear theory).



ODDC case, Moll et al, 2017

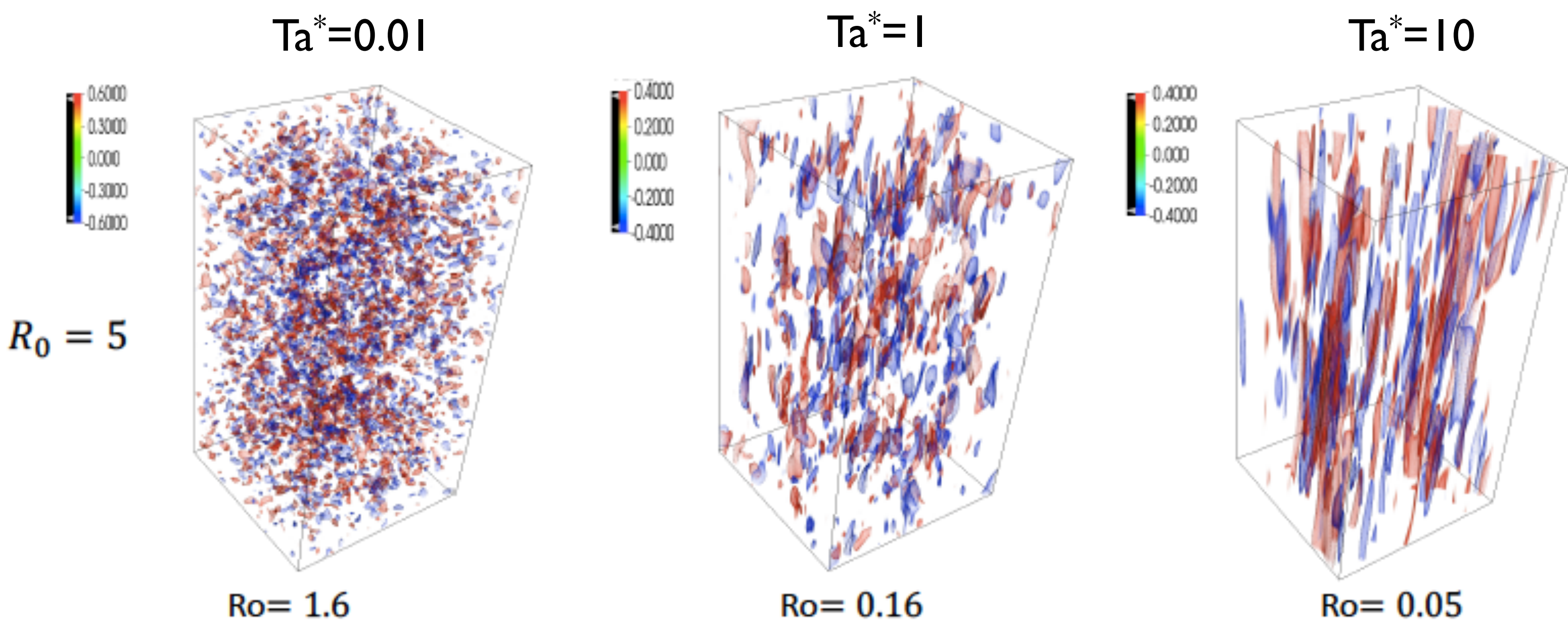
Beyond linear theory

Naïve expectation:

- Since fingers are aligned with axis of rotation, and since rotation tends to suppress motion perpendicular to axis of rotation, we expected the rotation to suppress *the shear instability* between the fingers
- If shear instability is harder to trigger, need higher vertical velocity to trigger it: we expect the velocity at saturation of the fingering instability to be larger
- This should cause enhanced transport by fingering convection, and may resolve the RGB problem.

Rotating fingering convection

- DNSs show that rotation causes the fingers to become elongated in the direction of rotation.



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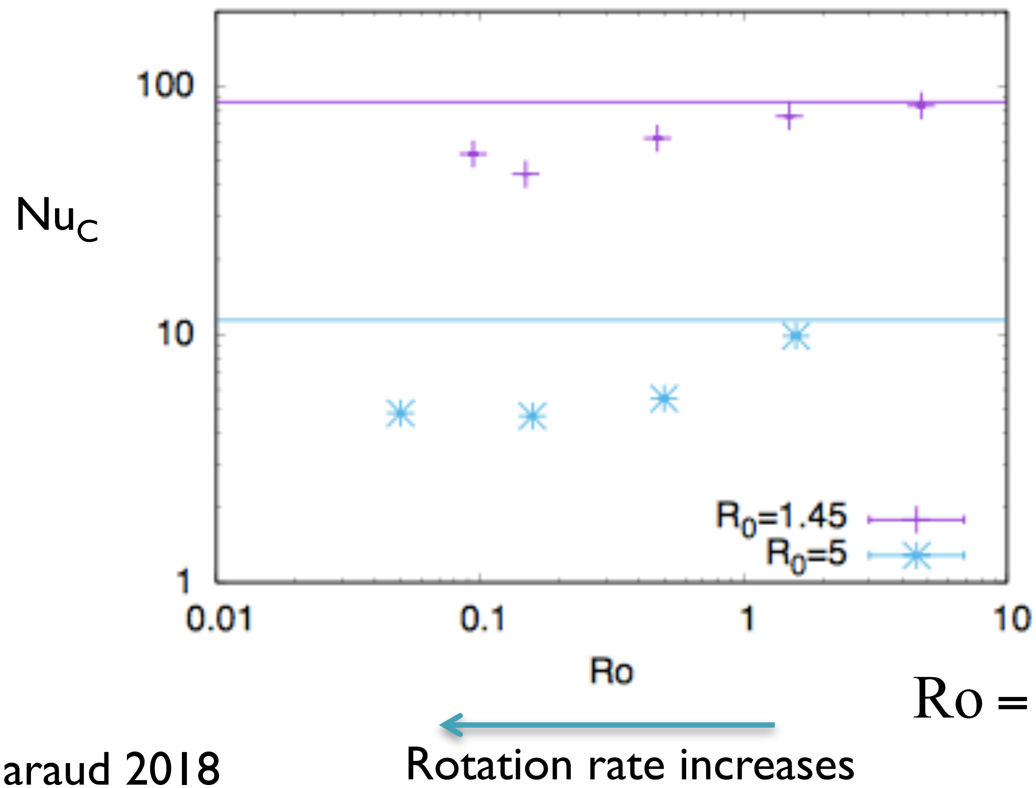
$\mathrm{Pr} = \tau = 0.1$

Rotating fingering convection

- But, we found that rotation typically *decreases* turbulent fluxes (but only a little).
- This was unexpected (and remains unexplained)



$$\mathrm{Ro} = Ta_*^{-1/2}\sqrt{\frac{\mathrm{Pr}}{R_0 - 1}}$$

Sengupta & Garaud 2018

The Rossby number of rotating DDC

- The effect of rotation on the *nonlinear* behavior of the instability depends on the Rossby number of the saturated turbulent flow.

$$\mathrm{Ro} = \frac{u_{rms}}{2\Omega l} = \frac{\hat{u}_{rms}}{\mathrm{Ta}^{1/2}\hat{l}}$$

- In general, u_{rms} would be difficult to predict a priori. However, in DDC we sometimes can.

- Recall in fingering convection, the value of l and u_{rms} at saturation for the non-rotating case can be predicted semi-analytically:

  - For large and moderate Pr :

$$\hat{l} \sim O(1), \hat{u}_{rms} \sim O(1) \rightarrow \mathrm{Ro} \approx Ta_*^{-1/2}$$

Ro >>1: Rotation is not very relevant in geophysics

The Rossby number of rotating DDC

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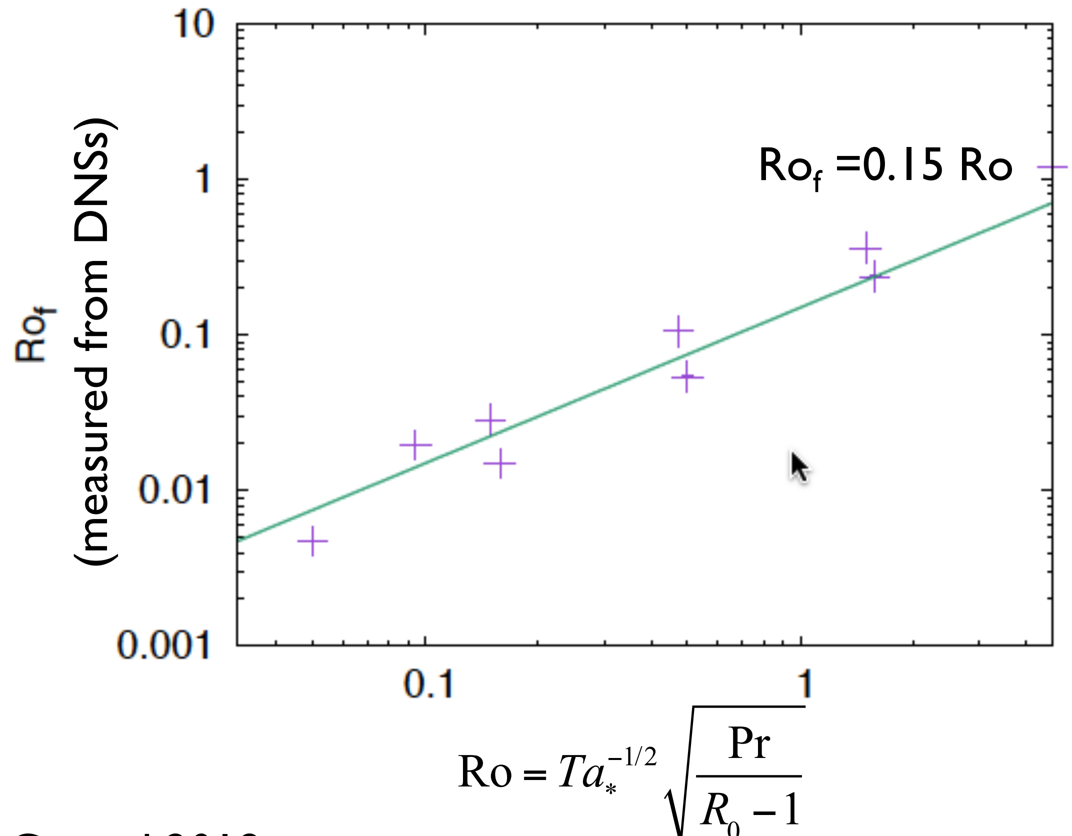
- Recall in fingering convection, the value of l and u_{rms} at saturation for the non-rotating case can be predicted semi-analytically:
  - For asymptotically low Pr (Brown et al. 2013):

$$\hat{l} \sim O(1), \hat{u}_{rms} \sim \sqrt{\frac{\mathrm{Pr}}{R_0 - 1}} \rightarrow \mathrm{Ro} \approx Ta_*^{-1/2}\sqrt{\frac{\mathrm{Pr}}{R_0 - 1}}$$

Ro <<1: Rotation is relevant in stars and giant planets

The Rossby number of rotating DDC

- The predicted Rossby number is a decent estimate for the actual Rossby number (within constant of order unity)



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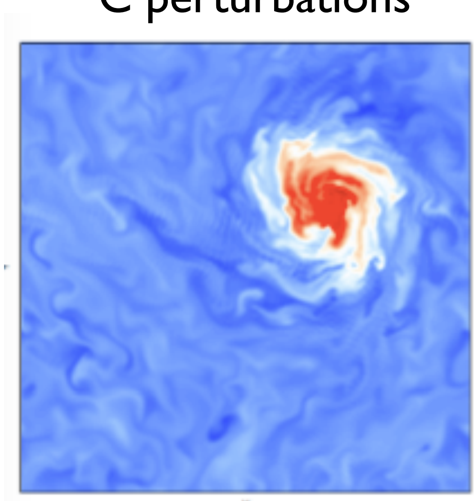
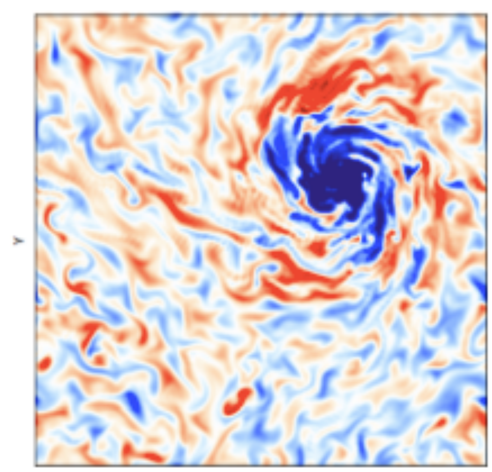
The formation of LSVs

- In some high Re, low Ro cases, a large-scale domain-filling cyclonic vortex forms
- The vortex core concentrates high-density material, which flows more rapidly.

C perturbations



Vertical velocity



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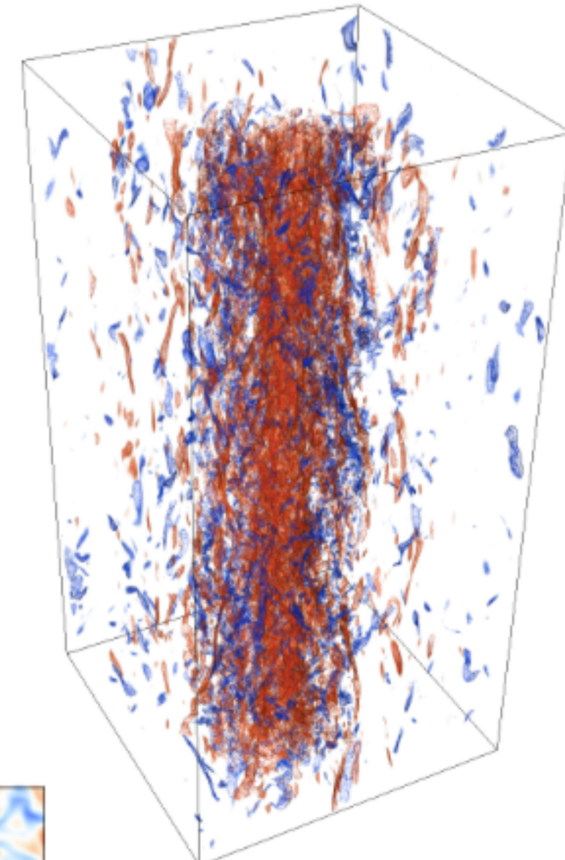

Vorticity

$$\mathrm{Pr} = \tau = 0.1$$

$$R_0 = 1.45$$

$$\mathrm{Ro} = 0.15$$

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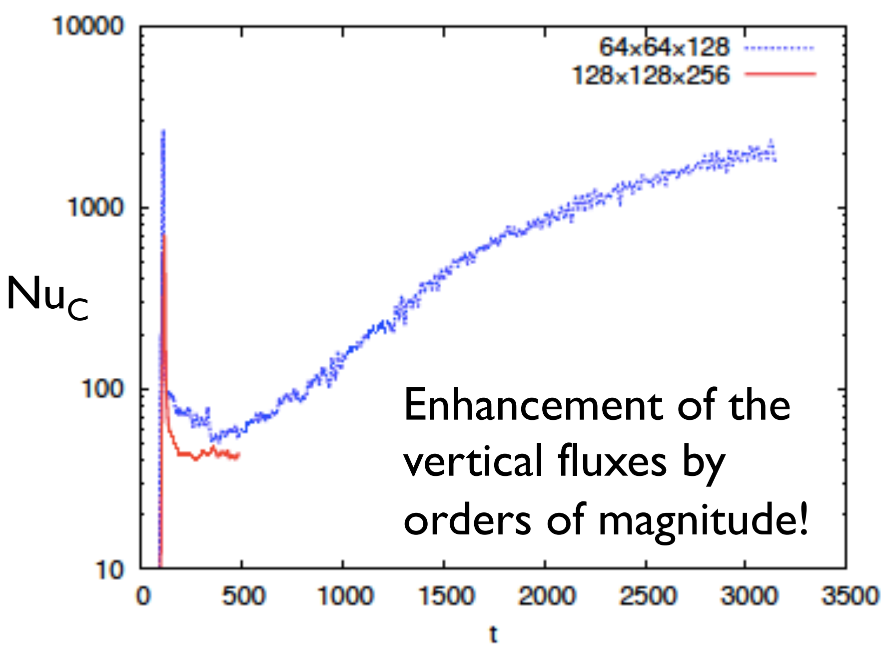
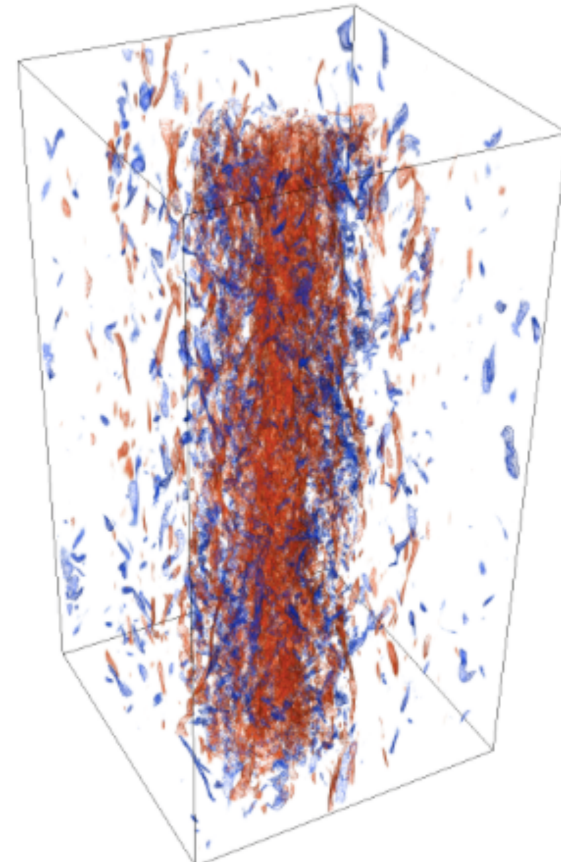




Vorticity

$$\mathrm{Pr} = \tau = 0.1$$

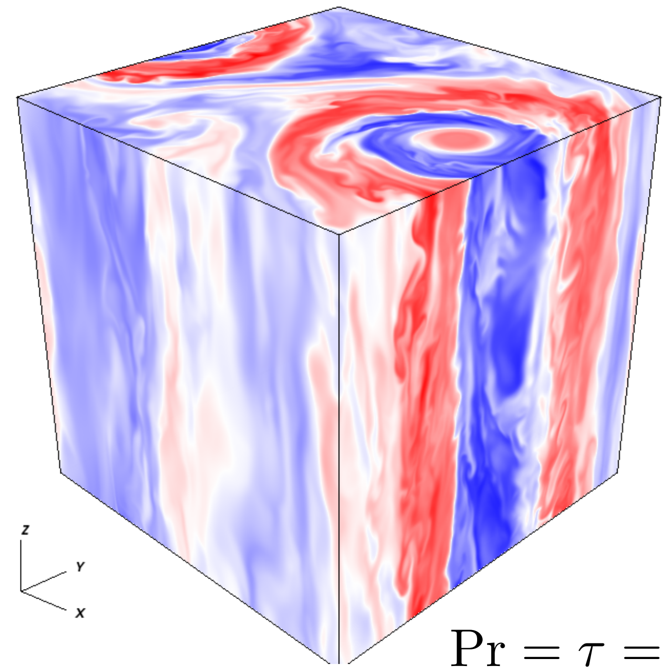
$$R_0 = 1.45$$

$$\mathrm{Ro} = 0.15$$

LSVs in rotating ODDC

- Similar large-scale vortices are observed in ODDC, with similar enhancement in transport compared with non-layered rotating ODDC.

- They are always cyclonic, typically have a complex multi-layer structure (high-C core, low-C sheath, high-C envelope)



$$\mathrm{Pr} = \tau = 0.1$$

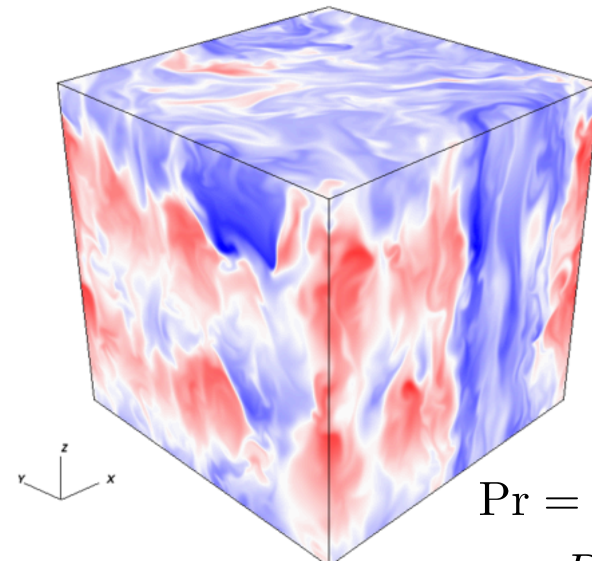
$$R_0 = 0.8$$

$$\mathrm{Ta}^* = 10$$

Moll & Garaud 2017

LSVs in rotating ODDC

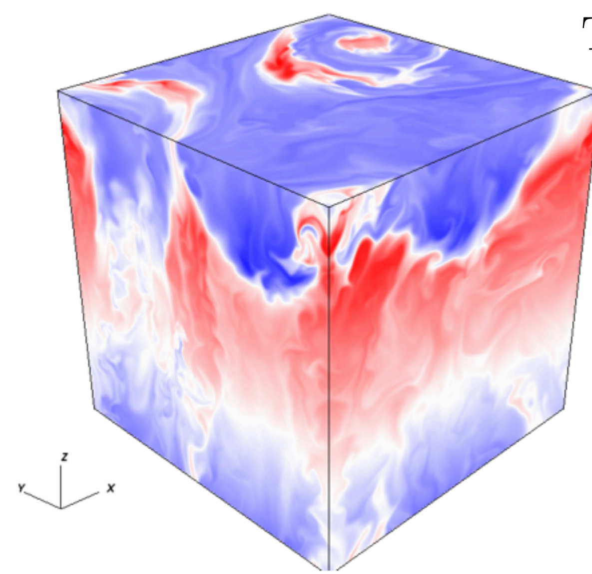
- Similar large-scale vortices are observed in ODDC, with similar enhancement in transport compared with non-layered rotating ODDC.
- They are always cyclonic, typically have a complex multi-layer structure (high-C core, low-C sheath, high-C envelope)
- They even form when layers are present!!



$\mathrm{Pr} = \tau = 0.1$

$R_0 = 0.8$

$\mathrm{Ta}^* = 1$



Moll & Garaud 2017

The formation of LSVs

- This is reminiscent of similar dynamics observed in rotating convection, & rotating stratified turbulence (Chan 2007, Kapyla et al. 2011, Julien et al. 2012, Rubio et al. 2014, Favier et al. 2014, Guervilly & Hughes 2014, Seshasayanan & Alexakis 2018, etc. ..)

- **In all these works, the vortices form at low Ro, high Re.**

- These works typically find:
  - 2D mode has power spectrum ~ k_h^{-3}
  - Energy provided into the 2D mode by combination 2D modes (local inverse cascade) and 3D modes (non-local inverse cascade).

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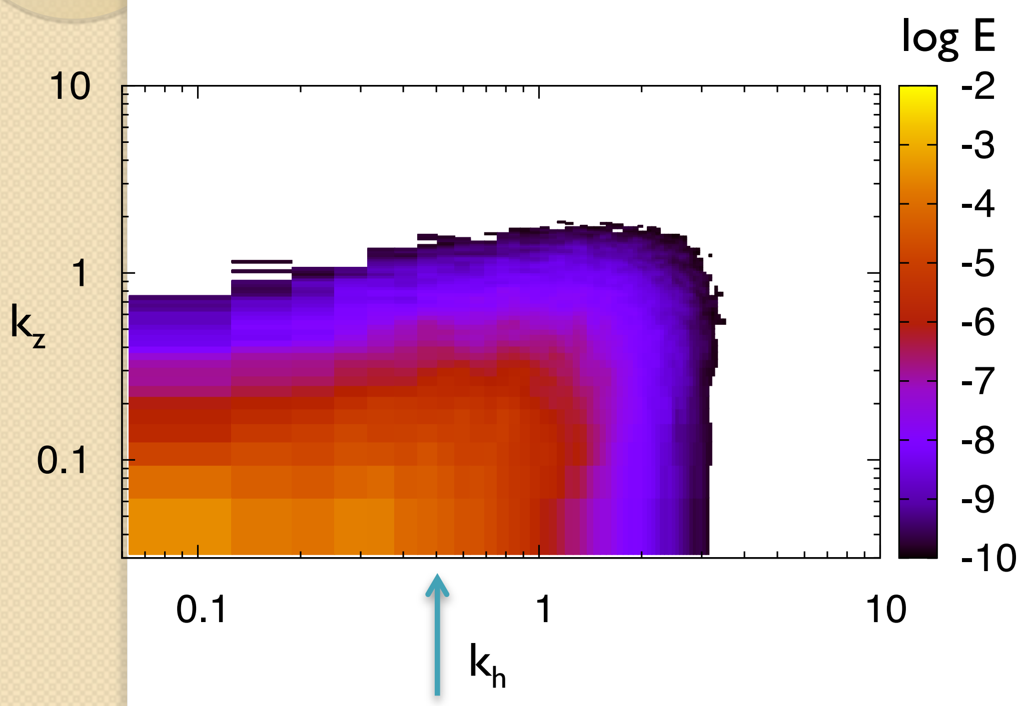
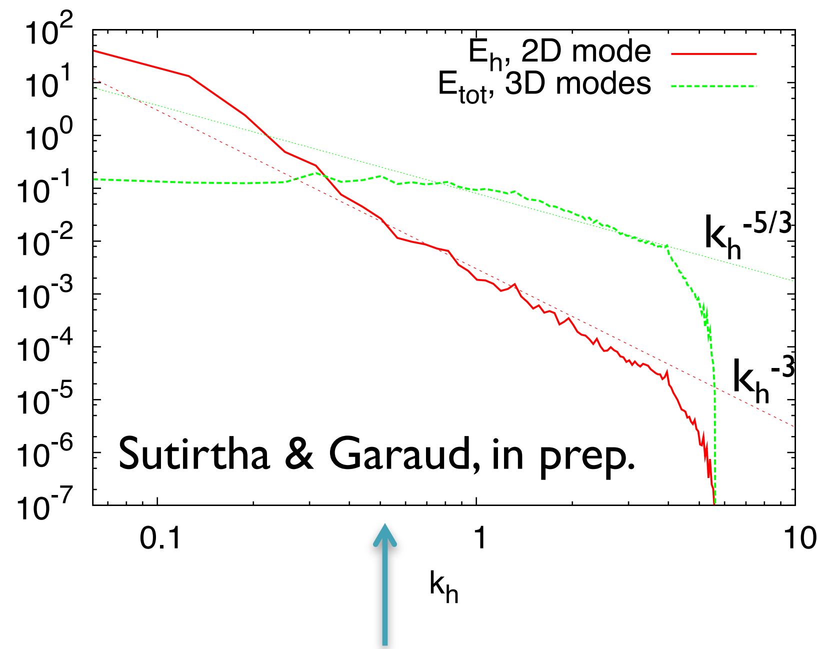
The formation of LSVs

- This is also true here.





Fingering fastest-growing mode

Prospect for mixing in RGB stars:

- In RGB stars, we expect that

$$\mathrm{Ro} \approx Ta_*^{-1/2}\sqrt{\frac{\mathrm{Pr}}{R_0-1}} \approx \frac{N}{\Omega}\sqrt{\frac{1}{R_0-1}} \sim 0.001-1$$

$$\mathrm{Re} \approx \frac{(2\pi)^2}{\sqrt{\mathrm{Pr}(R_0-1)}} \sim 10-100$$

for standard RGB parameter values (Pr ~ 10^{-6}, $R_0 \sim 10^3$)

- This is the correct parameter regime for LSV formation !

Maybe LSVs are the answer to the RBG missing mixing problem ?

Caveats:

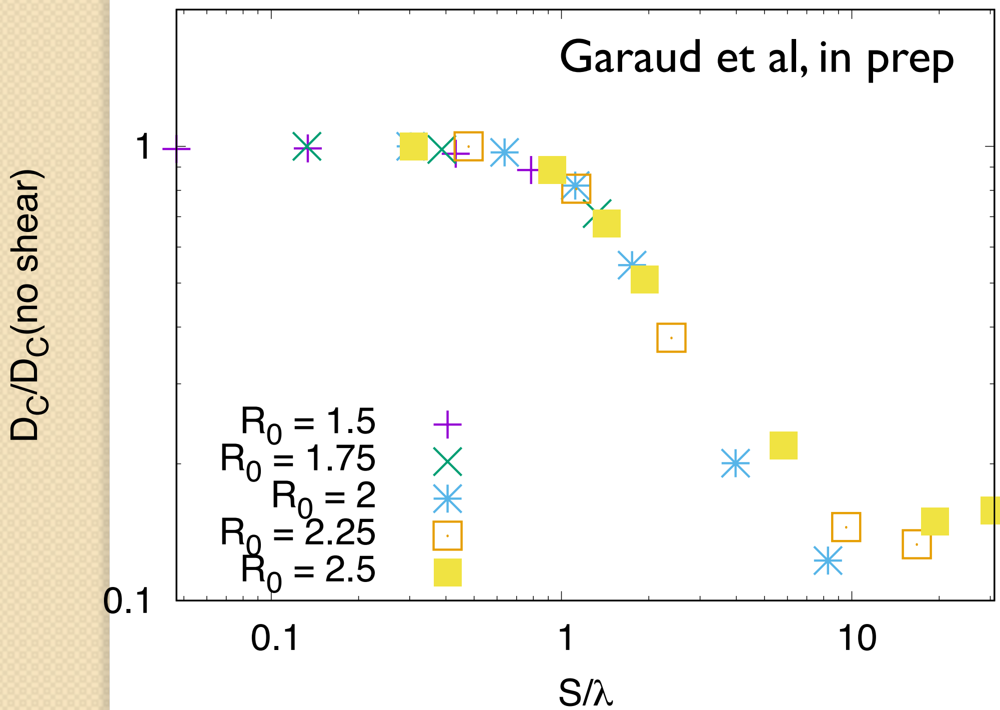
(1) We have not established whether they can exist away from the poles

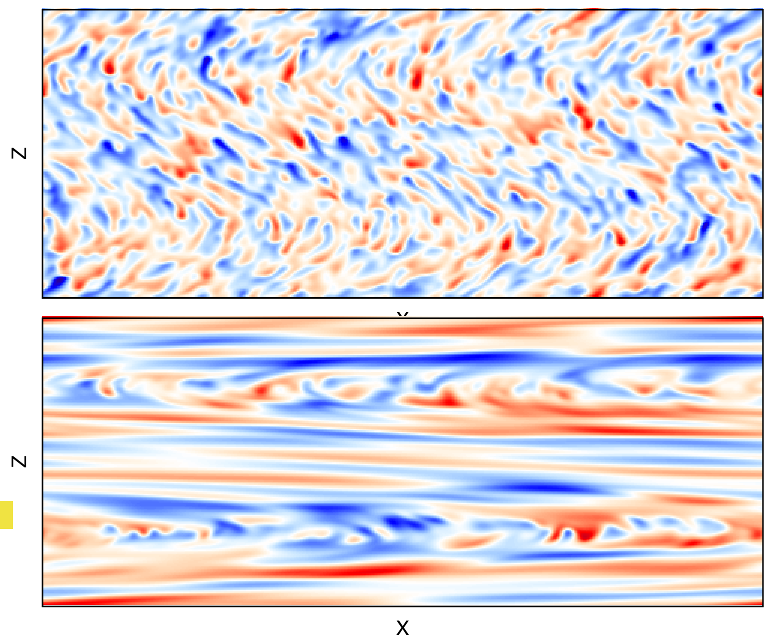
(2) Recent work (Julien et al. 2017) suggests they are aspect-ratio dependent

Shear, magnetic fields, …

- Shear and magnetic field significantly influence fingering convection as well…

Moderate shear tends to suppress fingering transport…





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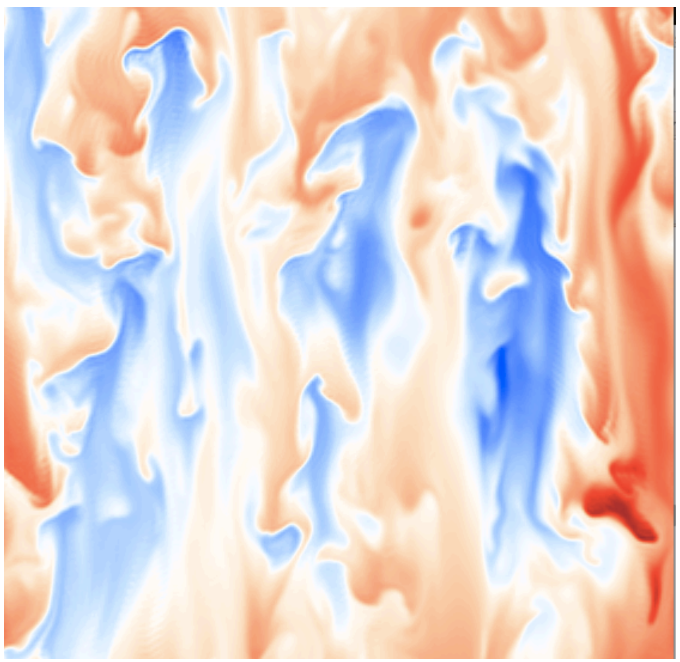
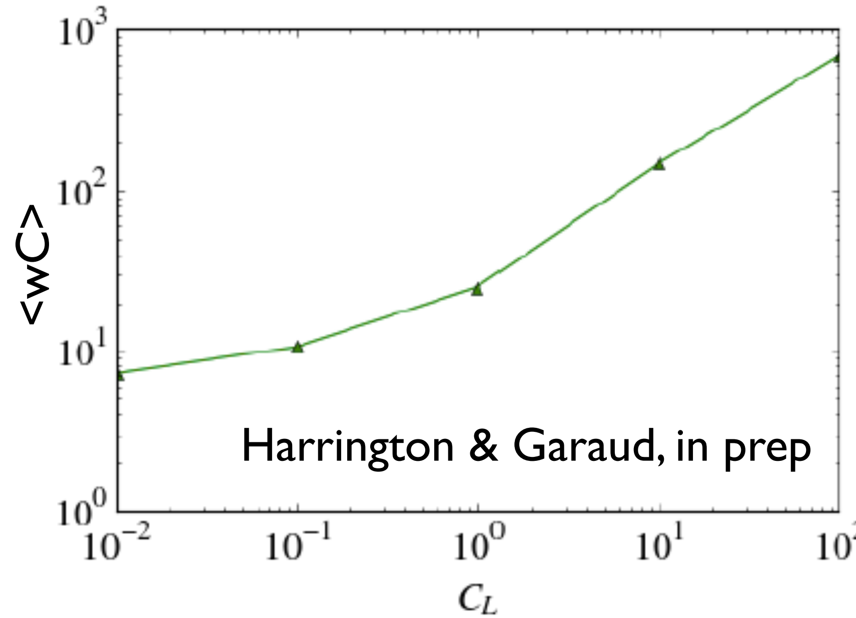


Magnetic fields tend to enhance it!



Double-diffusive instabilites are…

- Common in stars
- Have interesting dynamics
- Interact in non-trivial ways with other physical processes!