Double-diffusive processes in stellar astrophysics

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Plan

Lecture 1: General introduction

Lecture 2: Fingering convection

Lecture 3: ODDC (Semiconvection)

Lecture 4 (if time): The effect of rotation and shear

Double-Diffusive Convection at Low Prandtl Number

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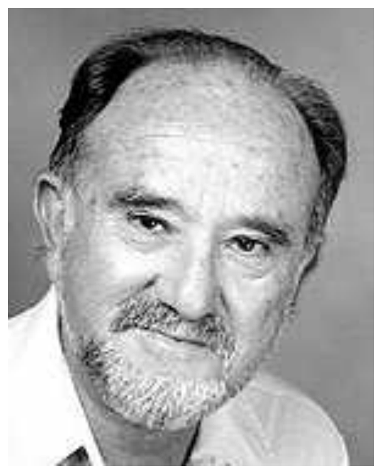
Lecture 1:
Double-diffusive instabilities, historical perspective.

The beginning

- Double-diffusive instabilities were first discovered in the context of oceanography in the 1950s/1960s, by a group of scientists at, or loosely affiliated with WHOI (Woods Hole Oceanographic Institution).

Henry Stommel

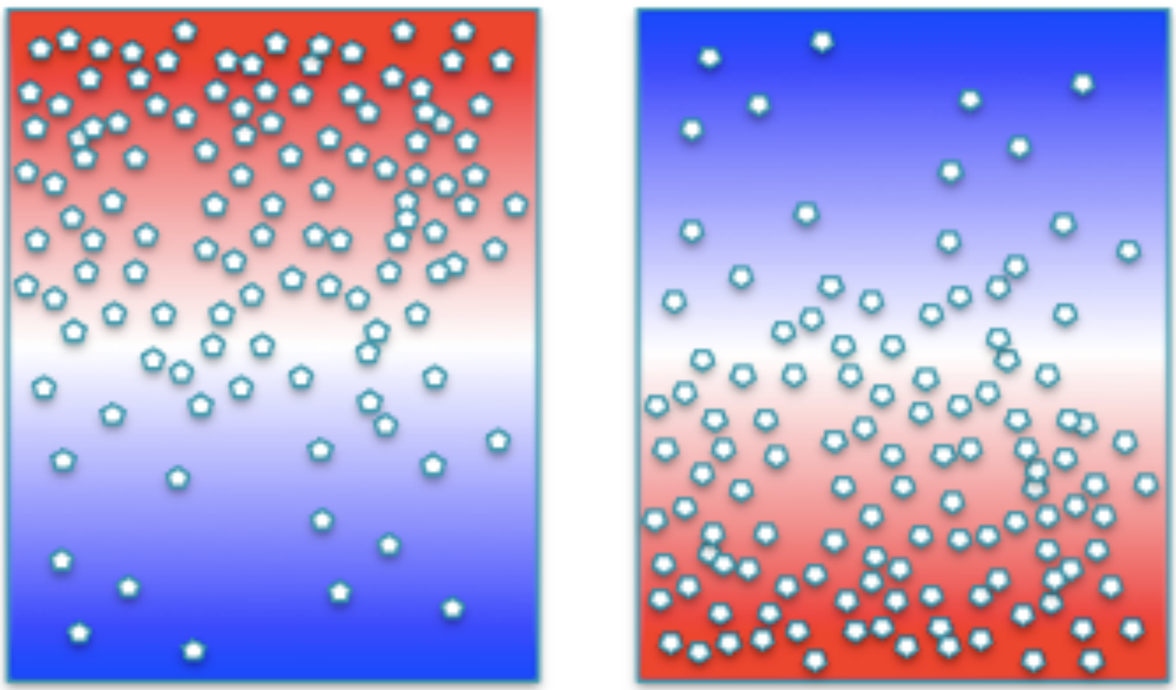


Melvin Stern

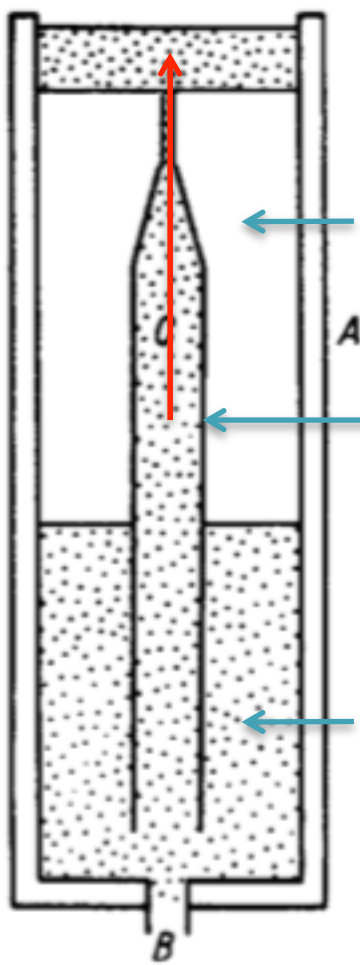
Louis Howard Melvin Stern Wilhelm Malkus

Doubly-stratified fluids

- Double-diffusive instabilities exist in doubly-stratified fluids
  - Stable temperature gradient with unstable composition
  - Unstable temperature gradient with stable composition



An oceanographical curiosity:
"the perpetual salt fountain" (Stommel et al, 1956)

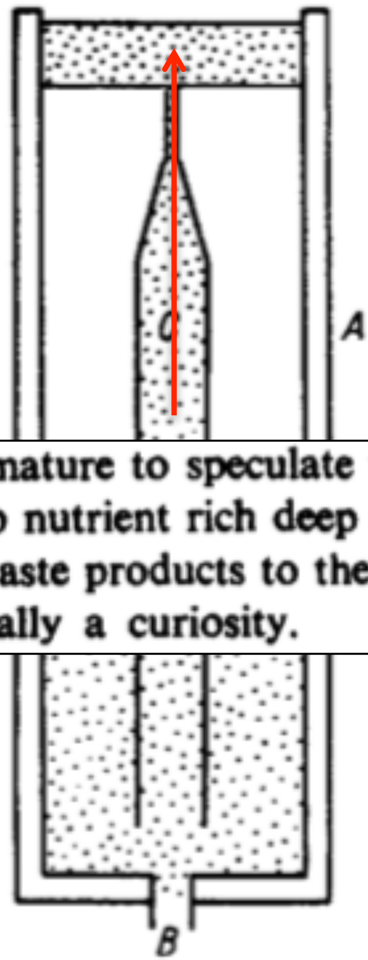


Hot salty water, total density is lower

The cold water in the tube warms up slightly, and becomes lower density than the salty hot water. A fountain gets created, that gives rise to a cold, fresh water layer on top

Cold fresh colored water, total density is higher

An oceanographical curiosity: “the perpetual salt fountain” (Stommel et al, 1956)



Although the attempt has not been made it is likely that in the Central North Atlantic with a tube 2,000 meters long, one might develop a pressure head of as much as two meters at the surface. A simple experi-

It seems premature to speculate upon the improbable practical importance of this phenomenon for pumping up nutrient rich deep water to the surface for fish-farming applications, or its inverse for removing waste products to the deep water. As a power source it is quite unpromising. Thus it remains essentially a curiosity.

The salt fountain and thermohaline convection (Stern, 1960)

STOMMEL, ARONS and BLANCHARD (1956) have described an "oceanographical curiosity" by noting that if a long vertical tube was lowered into the ocean, in such a manner that its bottom was exposed to cold fresh water and its top to warm saline water, a continuous motion could be maintained therein after priming the fountain. Their explanation is that the ascending (or descending) water in the tube would exchange heat but not salinity with the ambient ocean and would be accelerated due to its deficit in salt and density relative to fluid at the same level outside the tube. The purpose of this note, stemming from conversations with Henry Stommel, is to point out that in view of the great difference between the molecular diffusivity of salt ($K_S = 1.3 \times 10^{-5}$ cm^2 sec^{-1} for salinity of 35 ‰ at 20° C) and temperature ($K_T = 1.5 \times \times 10^{-3}$ cm^2 sec^{-1}) nature provides her own convective fountains. If a parcel of small

The tube is not actually needed!

The salt fountain and thermohaline convection (Stern, 1960)

Stern went on to lay out the linear stability analysis of thermohaline (fingering) convection. He also adds, in a footnote, the result for the case opposite stratifications (hot & salty below cold & fresh):

> [1] Compare the interesting case of a "solute" which is distributed so as to stabilize a super-adiabatic lapse rate of temperature. For exemple, if $f < 0$, $r > D > 1$ (i.e. $\beta_S < 0$, $\beta_T > 0$) the salt can stabilize the stationary mode and destabilize the oscillatory mode, thereby releasing the potential energy in the thermal stratification.

Both cases were then nicely unified in 1969:

On thermohaline convection with linear gradients

By P. G. BAINES AND A. E. GILL

Department of Applied Mathematics and Theoretical Physics,
University of Cambridge

Double-diffusive instabilities

- They are instabilities that occur when density depends on 2 independent components that diffuse at different rates. (In the ocean, temperature and salt)

- In this first lecture, I will review linear stability theory as derived by Baines & Gill, and will then move on to their applications in astrophysics.

Linear theory

The equation of state

- Each fluid can be characterized by an equation of state (EOS)
- Water is nearly incompressible, so ignore pressure contributions to EOS.
- If temperature variations and salt concentration variations are small, linearize EOS around a mean state "m" (Boussinesq approx.)

$$\rho(p,T,C) = \rho(T,C) = \rho(T_m,C_m) + \frac{\partial\rho}{\partial T}(T-T_m) + \frac{\partial\rho}{\partial C}(C-C_m)$$

$$\rightarrow \frac{\rho'}{\rho_m} = -\alpha T' + \beta C' \quad \text{where} \quad q' = q - q_m$$

$$\alpha = -\frac{1}{\rho_m}\frac{\partial\rho}{\partial T}, \beta = \frac{1}{\rho_m}\frac{\partial\rho}{\partial C}$$

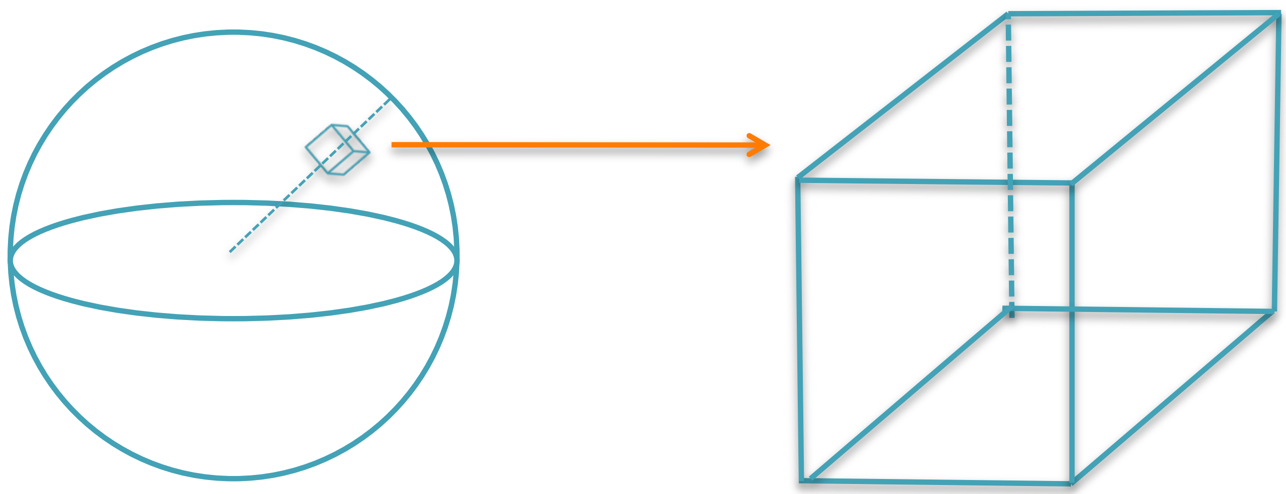
➔ For water at room temperature, α and β are positive so ρ decreases if T increases or C decreases

Mathematical modeling

Model considered:

- Assume **background** temperature or salinity profiles are linear (constant gradients T_{0z}, C_{0z})
- Let $T'(x,y,z,t) = zT_{0z} + \tilde{T}(x,y,z,t)$ and $C'(x,y,z,t) = zC_{0z} + \tilde{C}(x,y,z,t)$
- Assume that all **perturbations** are triply-periodic in domain (L_x,L_y,L_z)

- This enables us to study the phenomenon with little influence from boundaries.



Stability theory for single non-diffusive fluids

Let's first consider a single scalar (T or C)

(work on the board)

The criterion for instability for overturning convection is

$$T_{0z} < 0 \quad \text{or} \quad C_{0z} > 0$$

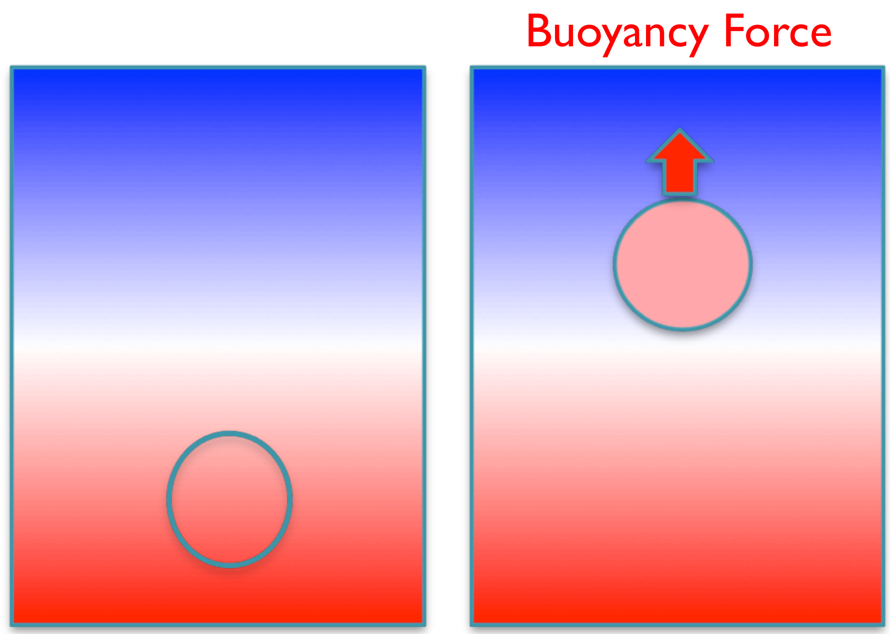
Overturning convection

- *Overturning* convection is a linear instability of stratified fluids with "top-heavy" density profiles, and occurs whenever $\rho_{0z} > 0$

Example: Water heated at the bottom is unstable to thermal convection

Instability criterion:

$$\rho_{0z} > 0 \Leftrightarrow T_{0z} < 0$$



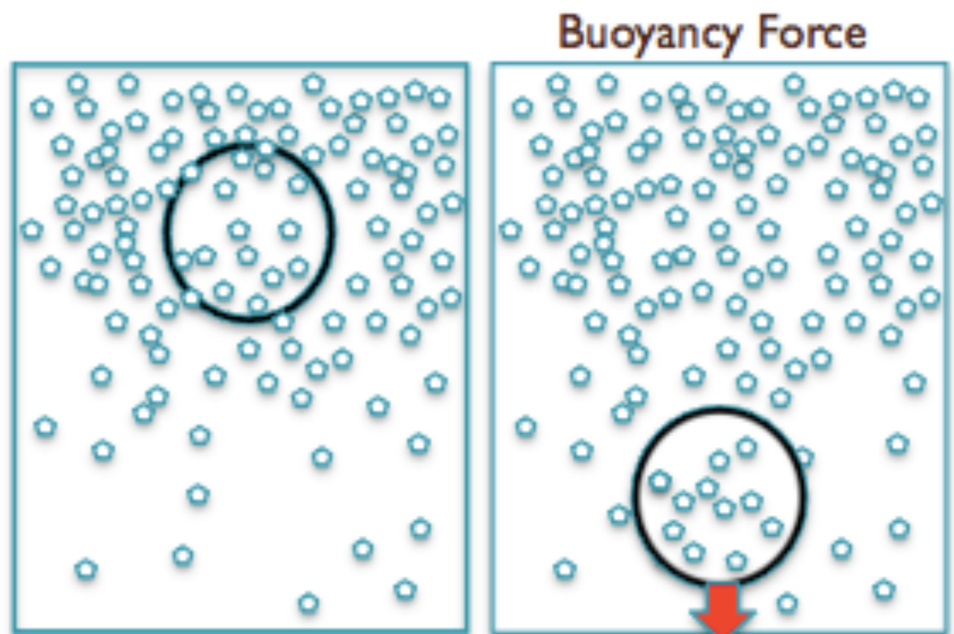
Overturning convection

- *Overturning* convection is a linear instability of stratified fluids with "top-heavy" density profiles, and occurs whenever $\rho_{0z} > 0$

Example: Overturning convection can occur in fluids with top-heavy composition.

Instability criterion:

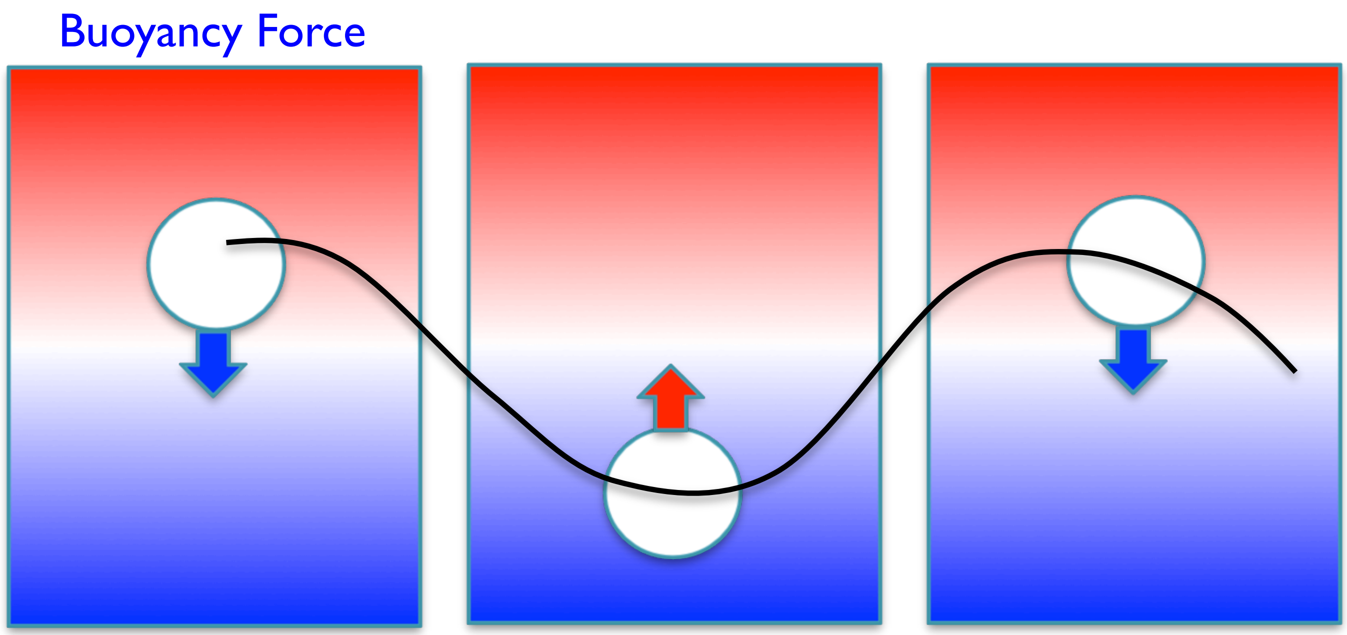
$$\rho_{0z} > 0 \Leftrightarrow C_{0z} > 0$$



Overturning convection vs. internal waves.

- Fluids that are stable to overturning convection ($\rho_{0z} < 0$) support internal gravity waves.

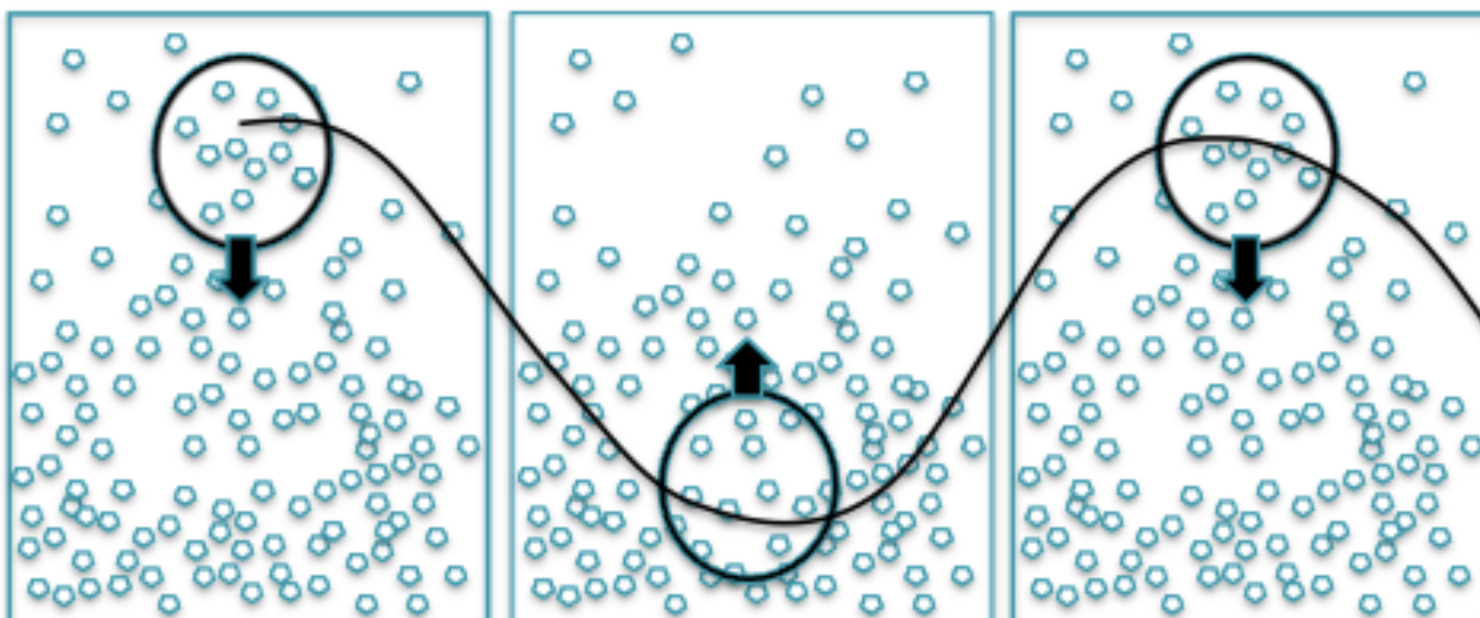
Example: Fluids heated from the top.



Overturning convection vs. internal waves.

- Fluids that are stable to overturning convection ($\rho_{0z} < 0$) support internal gravity waves.

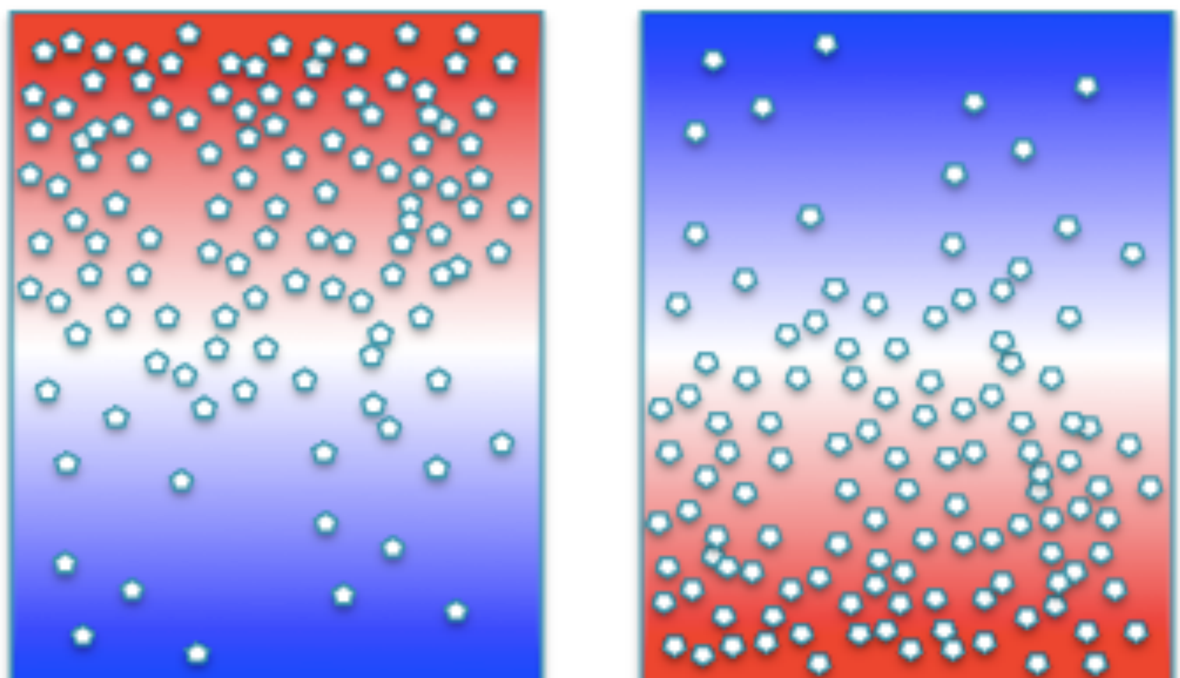
Example: Fluids with bottom-heavy composition



But what about the doubly-stratified case?

- Stable temperature gradient with unstable composition?
- Unstable temperature gradient with stable composition?

Instability criterion ?



Stability theory for non-diffusive doubly-stratified fluids.

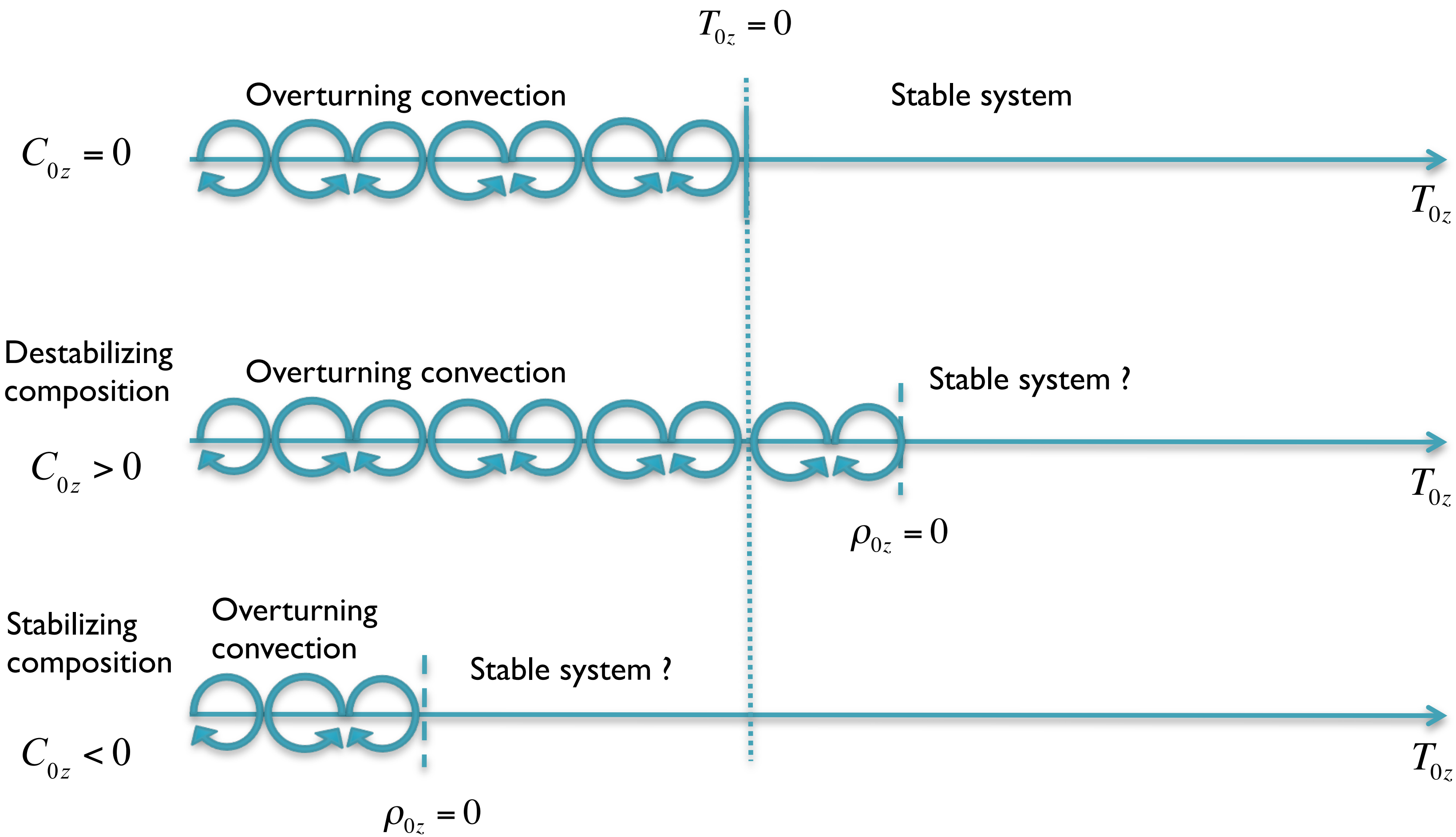
Combining the two scalars is not very difficult:

(work on the board)

The criterion for instability for overturning convection is

$$\rho_{0z} > 0 \quad \rightarrow \quad -\alpha T_{0z} + \beta C_{0z} > 0$$

Convection in non-diffusive multi-component fluids.

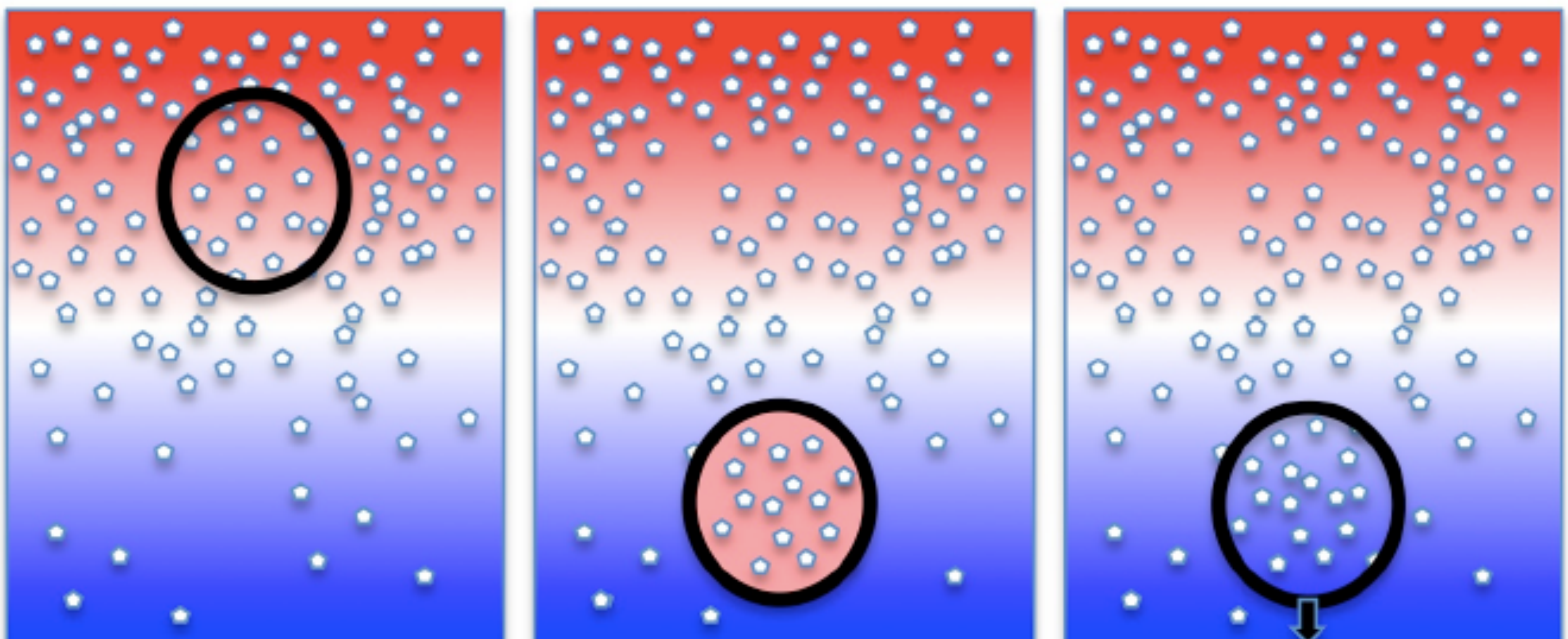


Double-diffusive convection

- However, temperature and composition usually diffuse at different rates
  - Salt diffuse about 100 times slower than heat in salt water
- When this is the case, new linear instabilities can occur *even in the case of stably stratified density profiles*.
- They are called "double-diffusive instabilities".

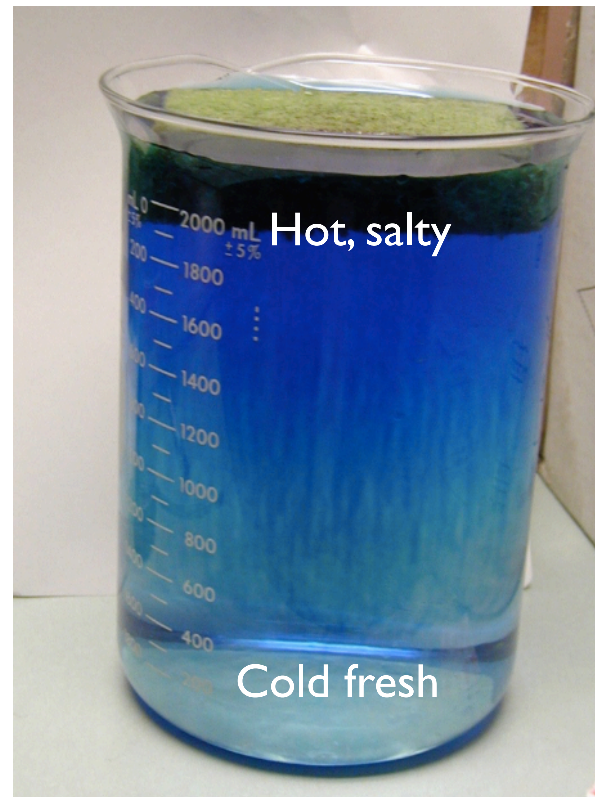
The fingering instability

Physical mechanism



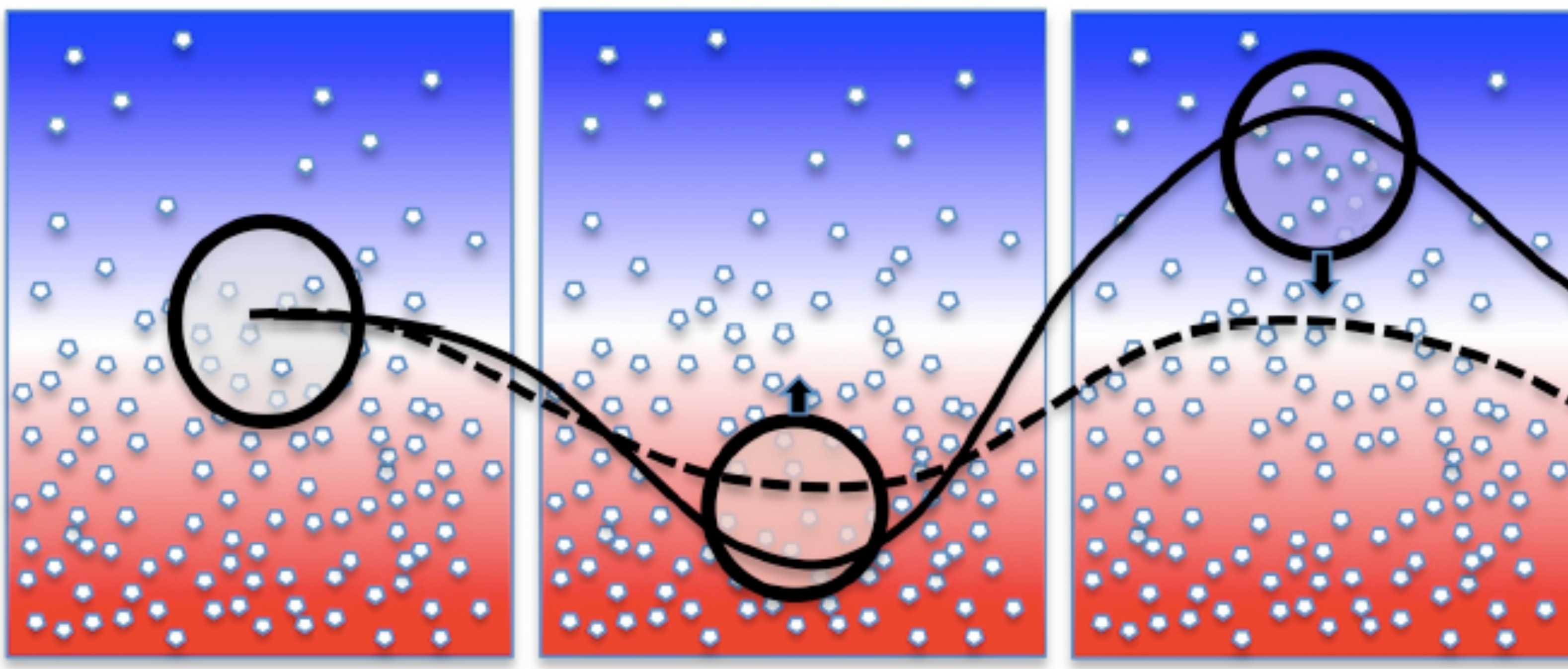
Fingering convection

Fingering convection is found in the tropical oceans, and manifests itself in the form of long, thin, "fingers" of hot/salty, cold/fresh plumes of water.



The oscillatory double-diffusive instability

Physical mechanism



ODDC

ODDC is in principle found in regions where cold, fresh water lies on top of hot, saltier water: the Arctic, and geothermally-active lakes.

Lake Kivu

Mathematical modeling

Governing equations (for incompressible salt water):

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho_m} - \frac{\rho}{\rho_m} g \hat{e}_z + \nu \nabla^2 \mathbf{u}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T + w T_{0z} = \kappa_T \nabla^2 T$$

$$\frac{\partial C}{\partial t} + \mathbf{u} \cdot \nabla C + w C_{0z} = \kappa_C \nabla^2 C$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\rho}{\rho_m} = -\alpha T + \beta C$$

Non-dimensionalization (work on the board)

Mathematical modeling

Governing non-dimensional equations:

$$\frac{1}{\Pr}\left(\frac{\partial \hat{\mathbf{u}}}{\partial t} + \hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}\right) = -\nabla\hat{p} + \left(\hat{T} - \hat{C}\right)\mathbf{e}_z + \nabla^2\hat{\mathbf{u}}$$

$$\frac{\partial \hat{T}}{\partial t} + \hat{\mathbf{u}}\cdot\nabla\hat{T} \pm \hat{w} = \nabla^2\hat{T}$$

$$\frac{\partial \hat{C}}{\partial t} + \hat{\mathbf{u}}\cdot\nabla\hat{C} \pm R_0^{-1}\hat{w} = \tau\nabla^2\hat{C}$$

$$\nabla\cdot\hat{\mathbf{u}} = 0$$

Governing parameters:

$$\Pr = \frac{\nu}{\kappa_T} \sim 7 \text{ for ocean}$$

$$\tau = \frac{\kappa_C}{\kappa_T} \sim 0.01 \text{ for ocean}$$

$$R_0 = \frac{\alpha T_{0z}}{\beta C_{0z}}$$

$$[l] = d = \left(\frac{\kappa_T \nu}{\alpha g \left|T_{0z}\right|}\right)^{1/4},$$

$$[t] = \frac{d^2}{\kappa_T}, \quad [T] = d\left|T_{0z}\right|, \quad [C] = \frac{\alpha}{\beta} d\left|T_{0z}\right|$$

Linear theory (basic instability)

Linear stability analysis:

- Assume all perturbations are of the form

$$\hat{q}(x,y,z,t) \propto e^{i\mathbf{k}\cdot\mathbf{x}+\lambda t}$$

- (Work on the board)

Linear theory (basic instability)

Linear stability analysis:

- Assume all perturbations are of the form

$$\hat{q}(x,y,z,t) \propto e^{i\mathbf{k}\cdot\mathbf{x}+\lambda t}$$

- Resulting equation for growth rate is a **cubic**

$$\lambda^3 + \lambda^2 |\mathbf{k}|^2 (1 + \Pr + \tau)$$

$$+\lambda\left[|\mathbf{k}|^4 (\Pr + \tau + \Pr\tau) \pm \Pr\frac{k_h^2}{|\mathbf{k}|^2}(1 - R_0^{-1})\right]$$

$$+\Pr\tau|\mathbf{k}|^6 \pm \Pr k_h^2(\tau - R_0^{-1}) = 0$$

Linear theory (basic instability)

General results:

- In both cases fastest-growing modes are always “elevator” $k_z = 0$ modes in this setup (see Radko 2013).
- Fingering regime: growth rate is real (pure exponential growth)
- ODDC regime: growth rate is complex (exponential + oscillatory)

Linear theory (fingering)

The necessary condition for instability depends on the **density ratio**

$$R_0 = \frac{\alpha T_{0z}}{\beta C_{0z}} = \frac{\text{Stabilizing temperature stratification}}{\text{Destabilizing salinity stratification}}$$

Instability to fingering occurs if

$$1 < R_0 < \frac{\kappa_T}{\kappa_C} = \frac{1}{\tau}$$

Marginal stability threshold ~ 100 in ocean,

Threshold for overturning convection

Linear theory (ODDC)

- The dynamics of the ODDC instability depends on the non-dimensional **inverse density ratio**

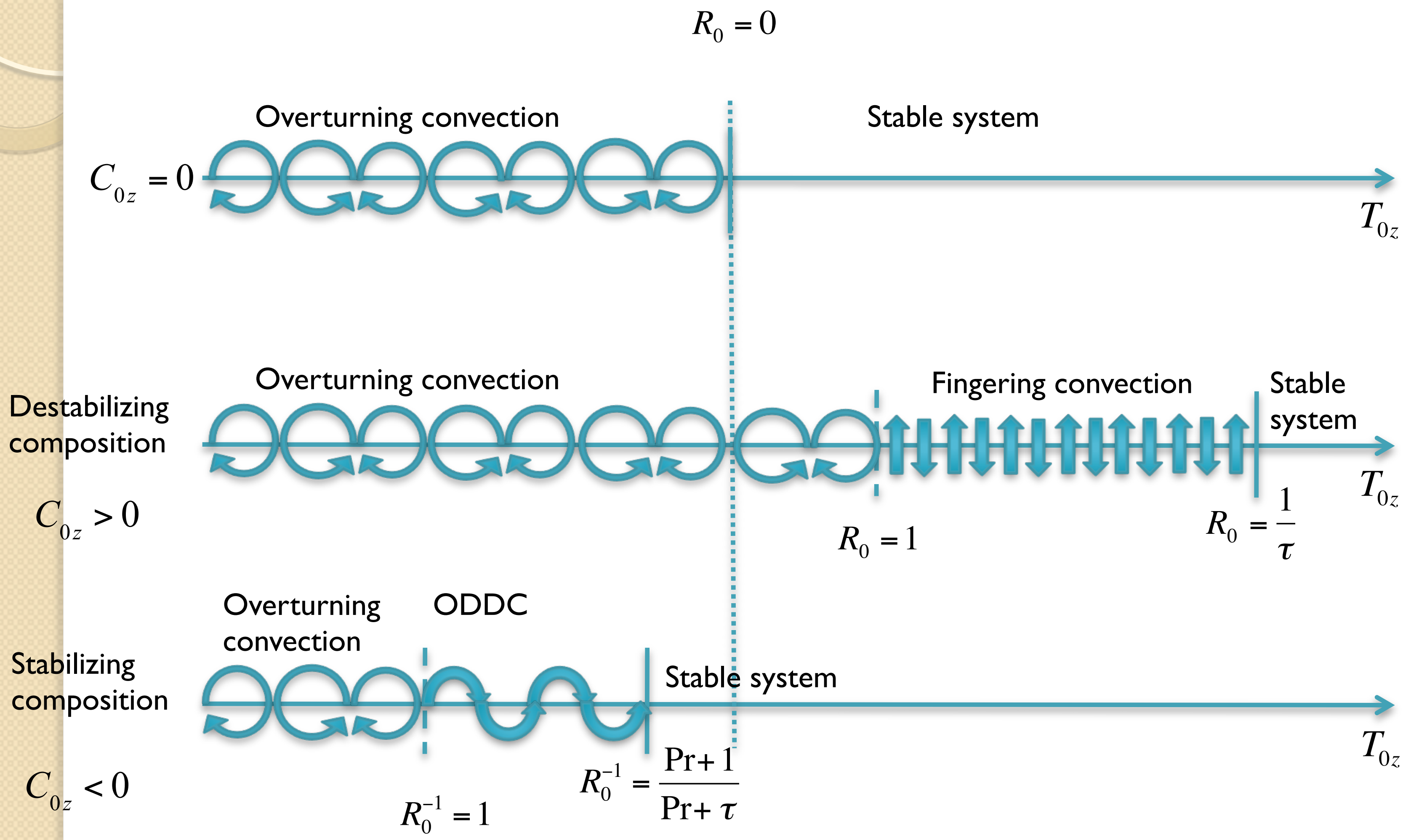
$$R_0^{-1} = \frac{\beta C_{0z}}{\alpha T_{0z}} = \frac{\text{Stabilizing salinity stratification}}{\text{Destabilizing temperature stratification}}$$

- **Linear instability to ODDC occurs if**

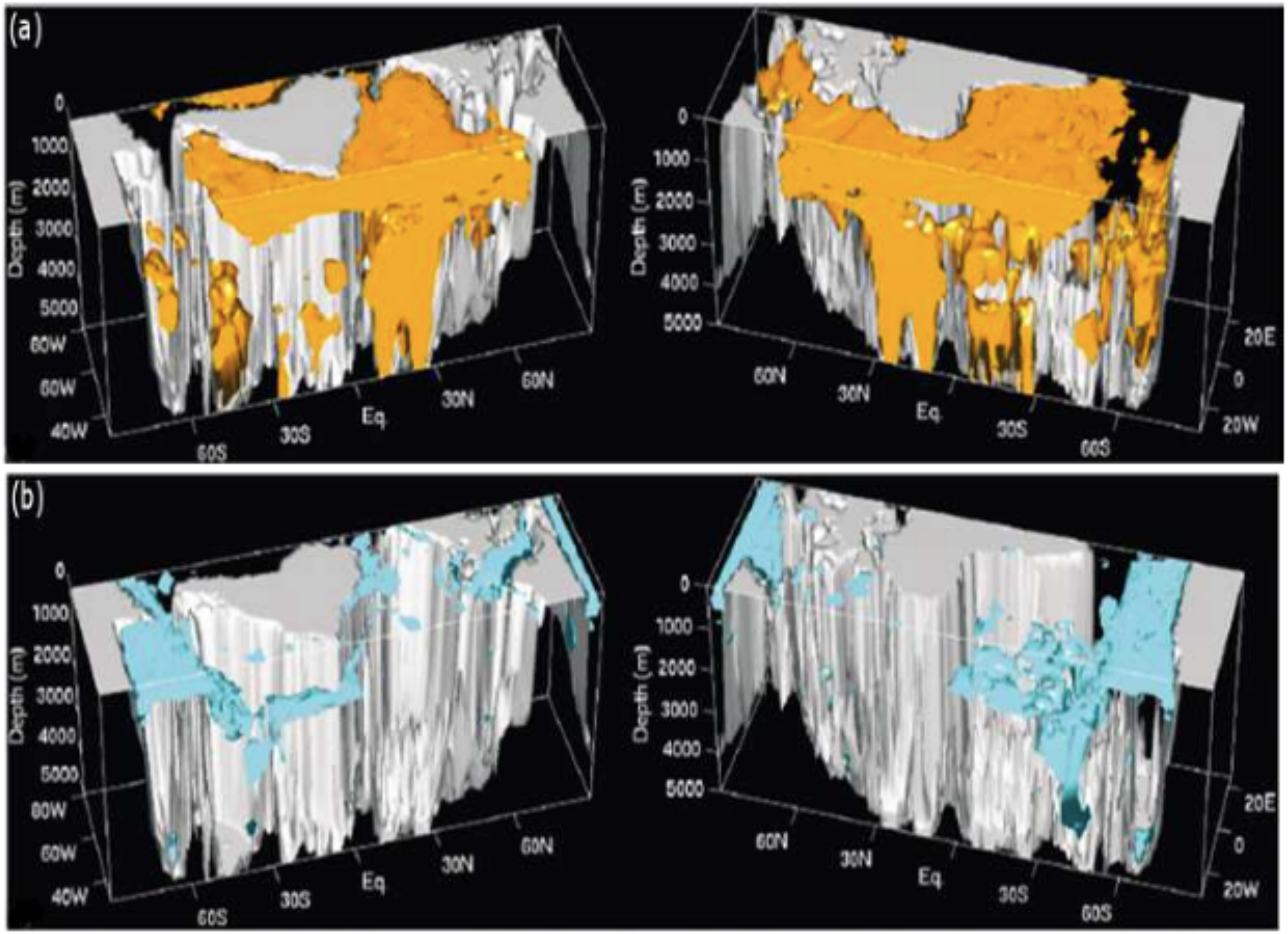
$$1 < R_0^{-1} < \frac{\nu + \kappa_T}{\nu + \kappa_C} = \frac{\mathrm{Pr} + 1}{\mathrm{Pr} + \tau}$$

Marginal stability threshold ~ 1.14 in ocean,

Linear theory (basic instability)



Regions of the ocean susceptible to DDC



From You (2002)

Double-diffusive instabilities in astrophysics

From the ocean to the stars

Main differences between salt water and the plasmas in stellar interiors are:

- C represents the concentration of a chemical species, rather than that of salt
- (Weak) compressibility
- Values of the parameters Pr and τ

The SV Boussinesq equations

Weak compressibility is addressed by using the Spiegel & Veronis (1960) Boussinesq equations for gases.

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla p}{\rho_m} - \frac{\rho}{\rho_m} g\mathbf{e}_z + \nu\nabla^2\mathbf{u}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T - wT_{0z}^{ad} = \kappa_T\nabla^2 T$$

$$\frac{\partial C}{\partial t} + \mathbf{u}\cdot\nabla C = \kappa_C\nabla^2 C$$

$$\nabla\cdot\mathbf{u} = 0$$

$$\frac{\rho}{\rho_m} = -\alpha T + \beta C$$

New term accounts for change in temperature due to slow expansion or contraction of parcel of gas as it adjusts to local pressure of surroundings. Note that

$$T_{0z}^{ad} = -\frac{g}{c_p}$$

Mathematical modeling

Governing non-dimensional equations:

$$\frac{1}{\Pr}\left(\frac{\partial \hat{\mathbf{u}}}{\partial t} + \hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}\right) = -\nabla\hat{p} + \left(\hat{T} - \hat{C}\right)\mathbf{e}_z + \nabla^2\hat{\mathbf{u}}$$

$$\frac{\partial \hat{T}}{\partial t} + \hat{\mathbf{u}}\cdot\nabla\hat{T} \pm \hat{w} = \nabla^2\hat{T}$$

$$\frac{\partial \hat{C}}{\partial t} + \hat{\mathbf{u}}\cdot\nabla\hat{C} \pm R_0^{-1}\hat{w} = \tau\nabla^2\hat{C}$$

$$\nabla\cdot\hat{\mathbf{u}} = 0$$

Exactly the same equations as in the oceanographic incompressible case!

Governing parameters:

$$\Pr = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_C}{\kappa_T}$$

$$R_0 = \frac{\alpha(T_{0z} - T_{0z}^{ad})}{\beta C_{0z}} = \frac{\delta\left(\nabla - \nabla_{ad}\right)}{\phi\nabla_\mu}$$

$$[l] = d = \left(\frac{\kappa_T \nu}{\alpha g\left|T_{0z} - T_{0z}^{ad}\right|}\right)^{1/4},$$

$$[t] = \frac{d^2}{\kappa_T}, \quad [T] = d\left|T_{0z} - T_{0z}^{ad}\right|, \quad [C] = \frac{\alpha}{\beta}d\left|T_{0z} - T_{0z}^{ad}\right|$$

Linear theory (basic instability)

- The dynamics of DD instabilities depends on the non-dimensional **density ratio**

$$R_0 = \frac{\nabla - \nabla_{ad}}{\frac{\phi}{\delta}\nabla_{\mu}}$$

R_0 = 1 corresponds to Ledoux criterion
R_0 = 0 corresponds to Schwartzschild criterion

- **Fingering case:** $R_0 = \frac{\text{Stabilizing gradient}}{\text{Destabilizing gradient}}$

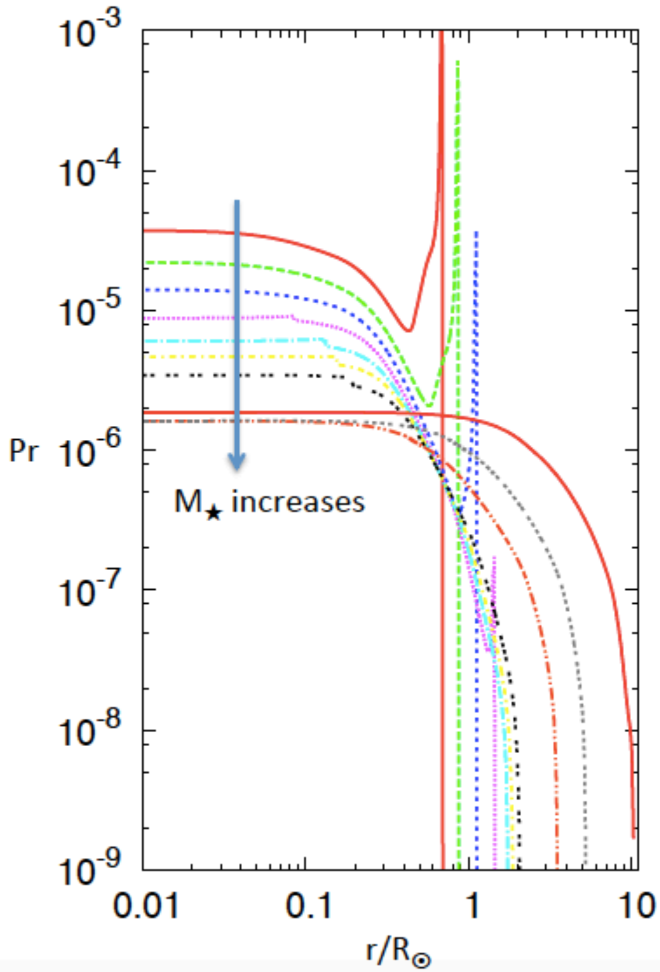
  Instability range: $1 < R_0 < \frac{\kappa_T}{\kappa_C} = \frac{1}{\tau}$

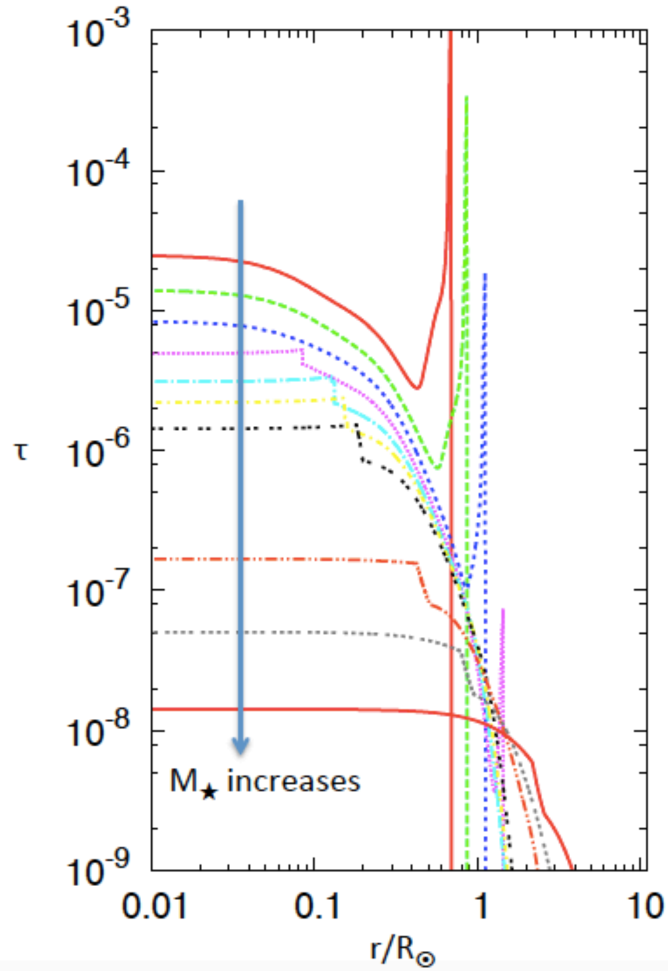
- **Oscillatory case:** $R_0 = \frac{\text{Detabilizing gradient}}{\text{Stabilizing gradient}}$

  Instability range: $\frac{\Pr + \tau}{\Pr + 1} = \frac{\nu + \kappa_C}{\nu + \kappa_T} < R_0 < 1$

Parameter values in stars

- Pr and τ are typically very small in stellar interiors ($\sim 10^{-6}$)

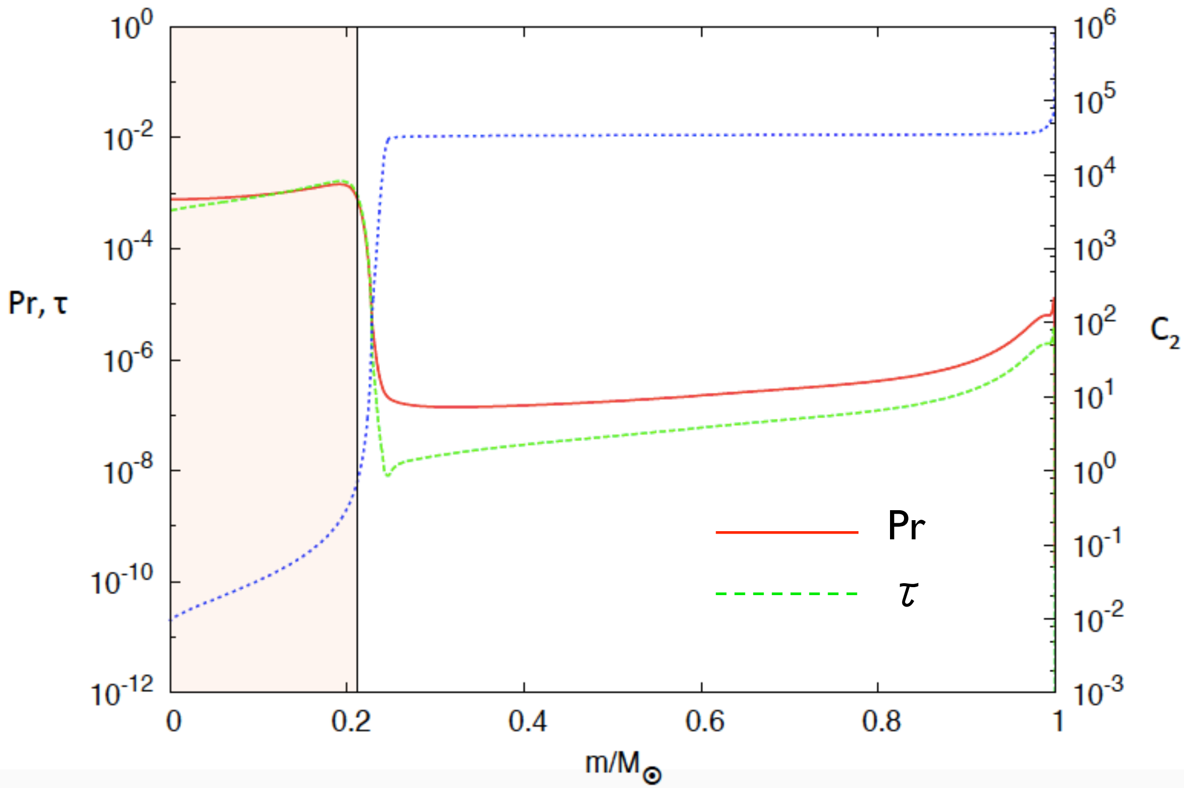




Pr and τ in MS stars of various masses (from Garaud et al. 2015)

Parameter values in stars

- Pr and τ are typically very small in stellar interiors ($\sim 10^{-6}$)
- With the exception of degenerate regions where electron conduction becomes important (e.g. RGB stars, WD stars)



Pr and τ in a $1M_{sun}$ RGB star star just before the luminosity bump (from Garaud et al. 2015)

Linear theory (basic instability)

- The dynamics of DD instabilities depends on the non-dimensional **density ratio**

$$R_0 = \frac{\nabla - \nabla_{ad}}{\frac{\phi}{\delta}\nabla_\mu}$$

R_0 = 1 corresponds to Ledoux criterion
R_0 = 0 corresponds to Schwartzschild criterion

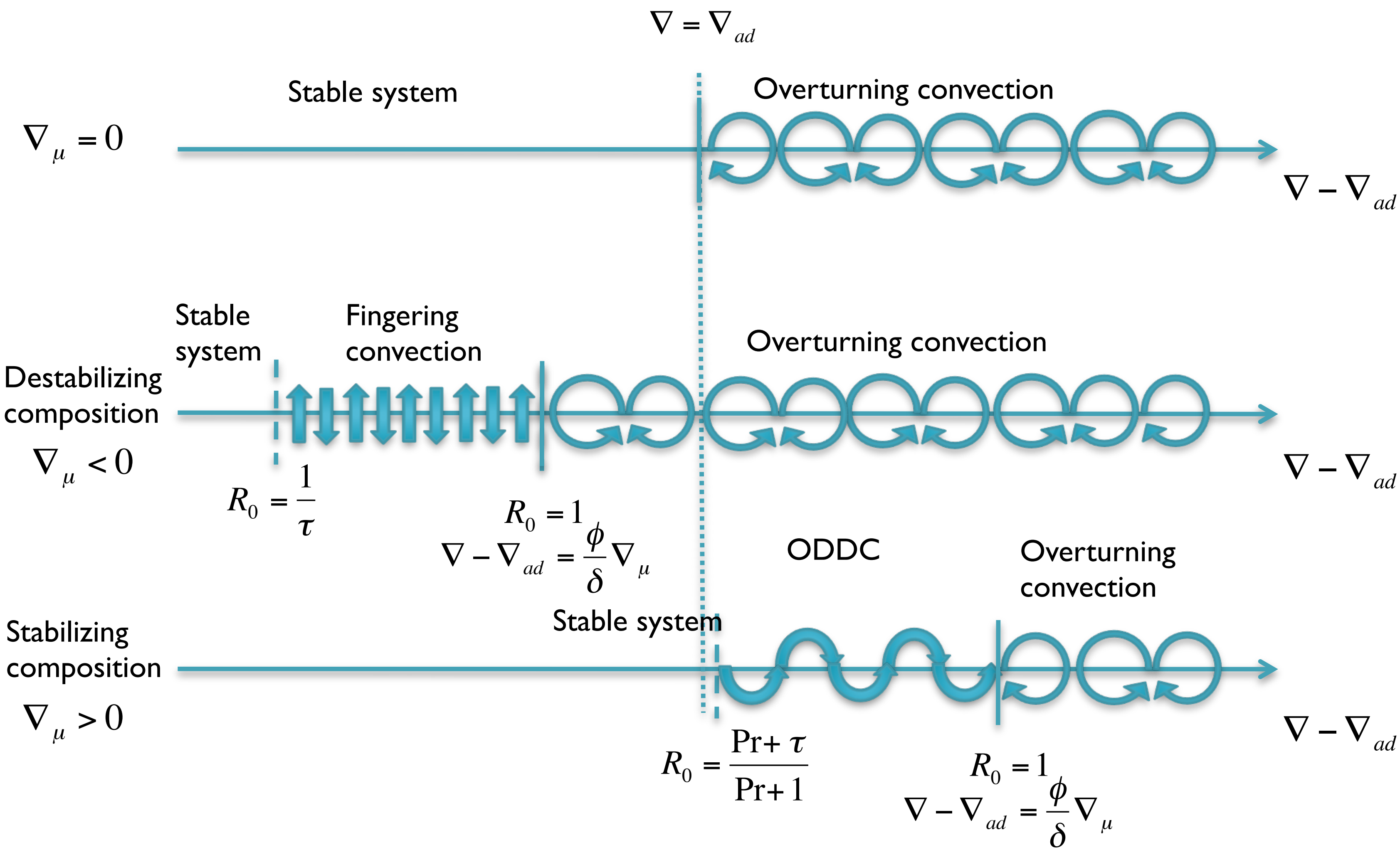
- **Fingering case:** $R_0 = \frac{\text{Stabilizing gradient}}{\text{Destabilizing gradient}}$

Instability range: $1 < R_0 < \frac{1}{\tau}$ ← >>1 in stars

- **Oscillatory case:** $R_0 = \frac{\text{Detabilizing gradient}}{\text{Stabilizing gradient}}$

Instability range: $\frac{\mathrm{Pr}+\tau}{\mathrm{Pr}+1} < R_0 < 1$ ← <<1 in stars

Convection in multi-component fluids.



Fingering convection in astrophysics

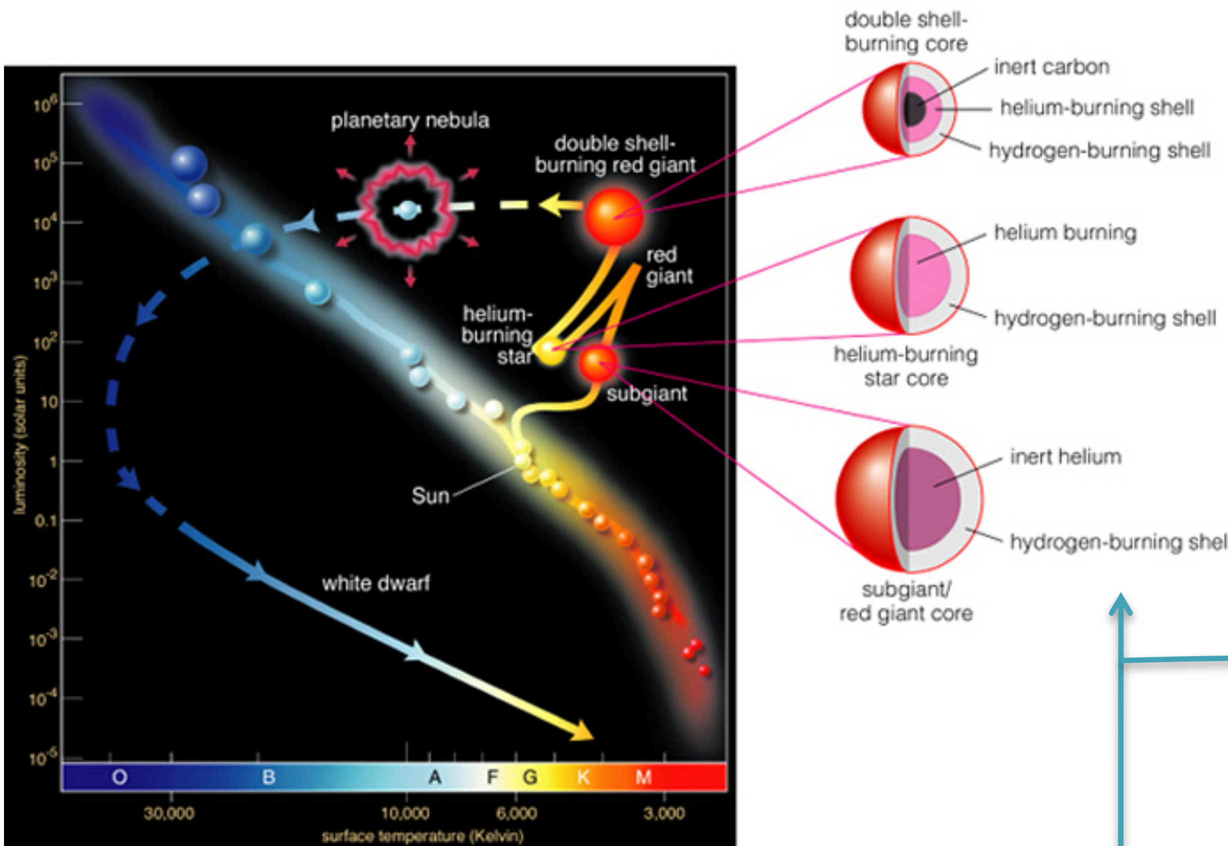
Fingering convection in astrophysics

Fingering convection found a number of situations with stable potential temperature (entropy) gradient (aka radiative zones) and unstable mean molecular weight gradient.

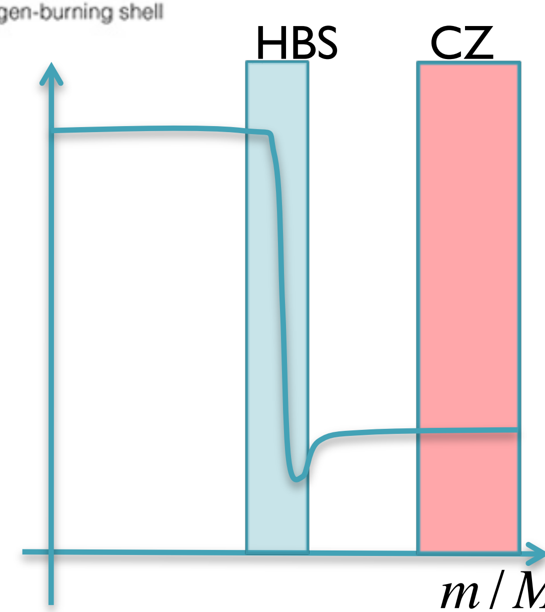
This can happen with e.g.

- Shell burning in RGB stars.
- Accretion of higher-mu material on top of the star
- Thin element layers in intermediate-mass stars

Fingering in RGB stars



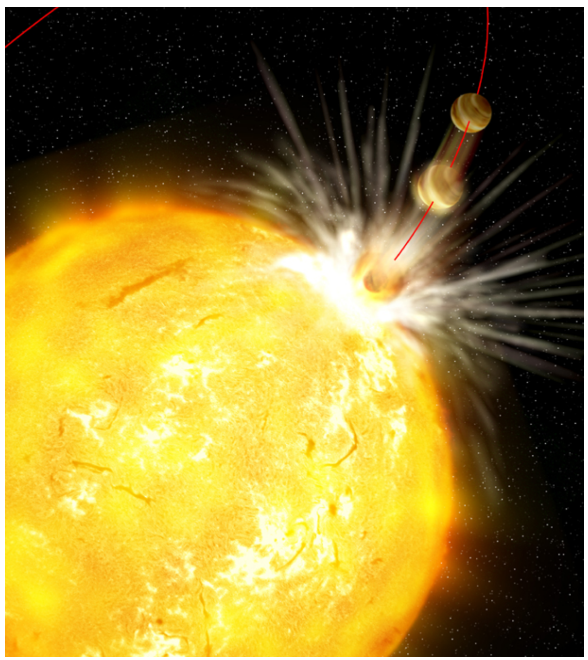
Inverse mu-gradient forms at edge of HBS

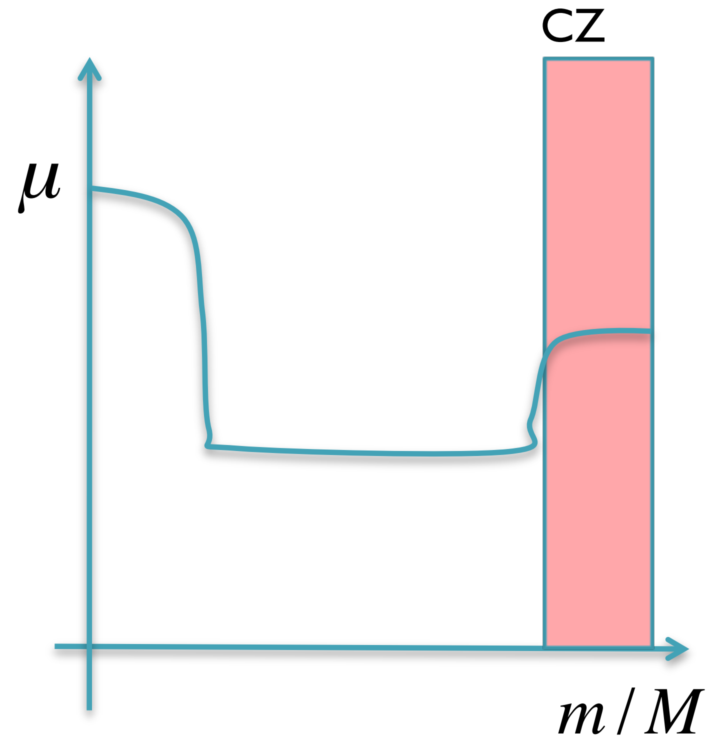


Ulrich 1972, Charbonnel & Zahn 2007, Denissenkov et al. 2008, 2010,2011, Cantiello & Langer 2011, Traxler et al. 2011, Lagarde et al. 2011, Wachlin et el. 2014

Accretion of high-mu material

Planetary infall on top of a star (MS star, WD stars), causes an inverse mu-gradient just below the surface (or below the surface CZ if there is one).

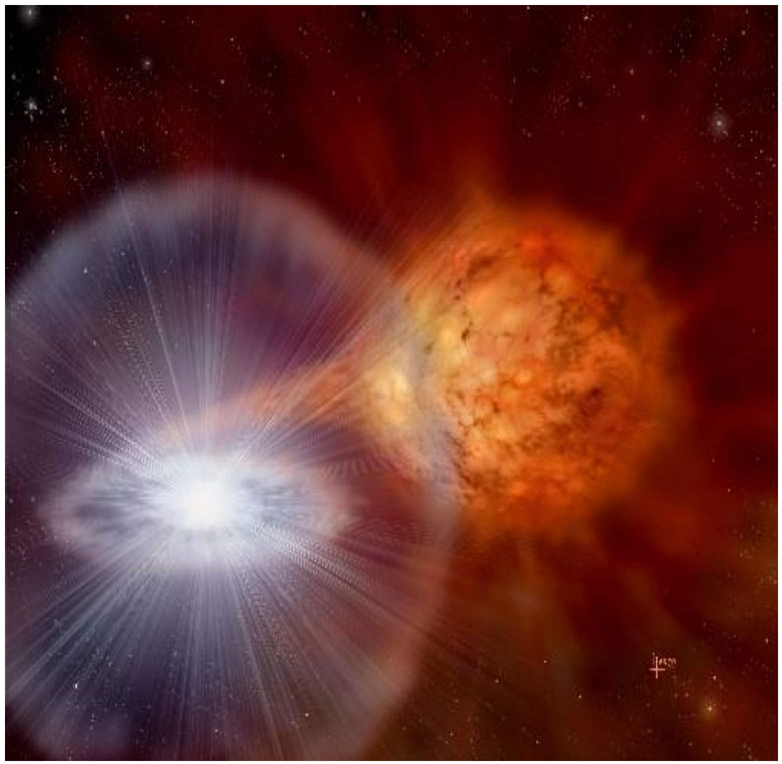




Vauclair 2004, Garaud 2011, Theado & Vauclair 2012, Deal et al. 2013, 2015, Wyatt et al. 2014, Wachlin et al. 2017, Bauer et al. 2018

Accretion of high-mu material

Similar inverse mu-gradients can arise because of accretion from more evolved companion star.

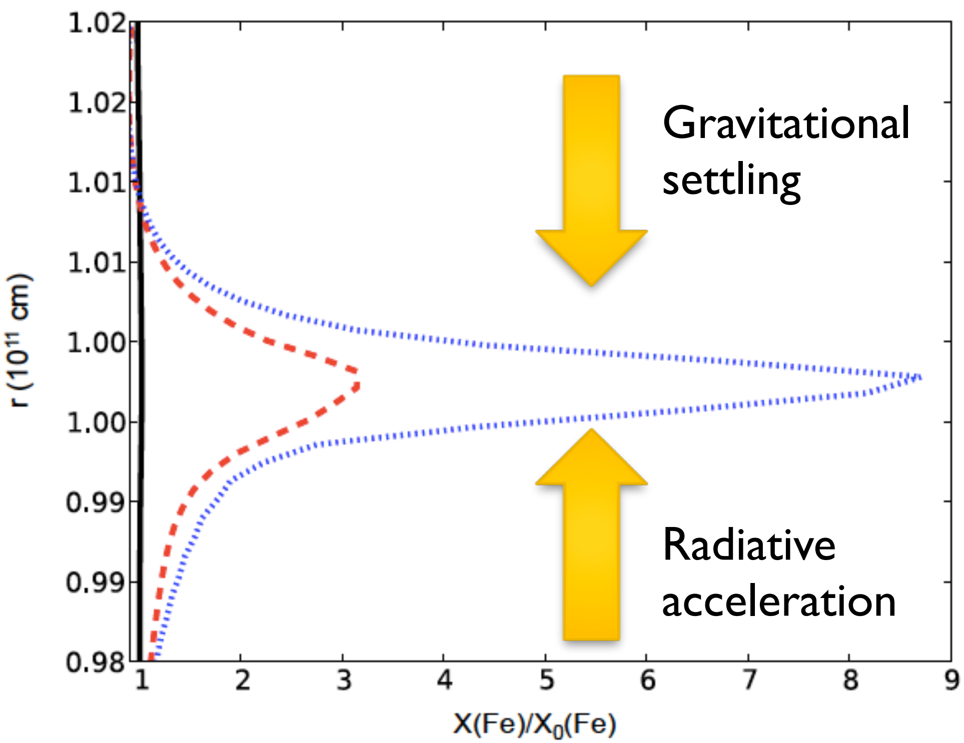


Marks & Sarna 1998,
Stancliffe et al. 2007;
Thompson et al. 2008;
Stancliffe 2009, …

In all these cases, the main question asked is "How much mixing does the fingering instability cause?"

Element layers in stars

- Combination of radiative levitation and gravitational settling causes accumulation of some elements near their opacity peak (Richard et al. 2001, …)
- This can be fingering-unstable (Theado et al. 2009, …)



Evolution of X(Fe) in a 1.7 M_{sun} star over 30Myr, from Zemskova et al. 2014.

ODDC in stars

ODDC in astrophysics

ODDC is found in situations with unstable potential temperature (entropy) gradient but unstable mu-gradient. These regions are Ledoux-stable but Schwarzschild-unstable. This situation is characteristic of so-called semi-convective regions in stars.

The terminology "semi-convection" was first introduced by Schwarzschild & Harm (1958). However, the connection with ODDC was only realized later (Kato 1966).

ODDC regions are mostly found outside convective cores, when a strong mu-gradient is present.

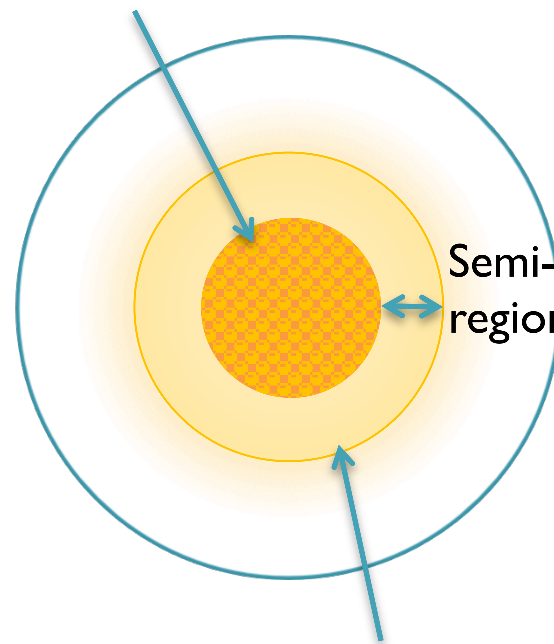
Mu-gradients outside convective cores

There are typically two mass-ranges in which they occur on the MS:

- Intermediate-mass stars (1-2 Msun)
- High-mass stars (>10 Msun)

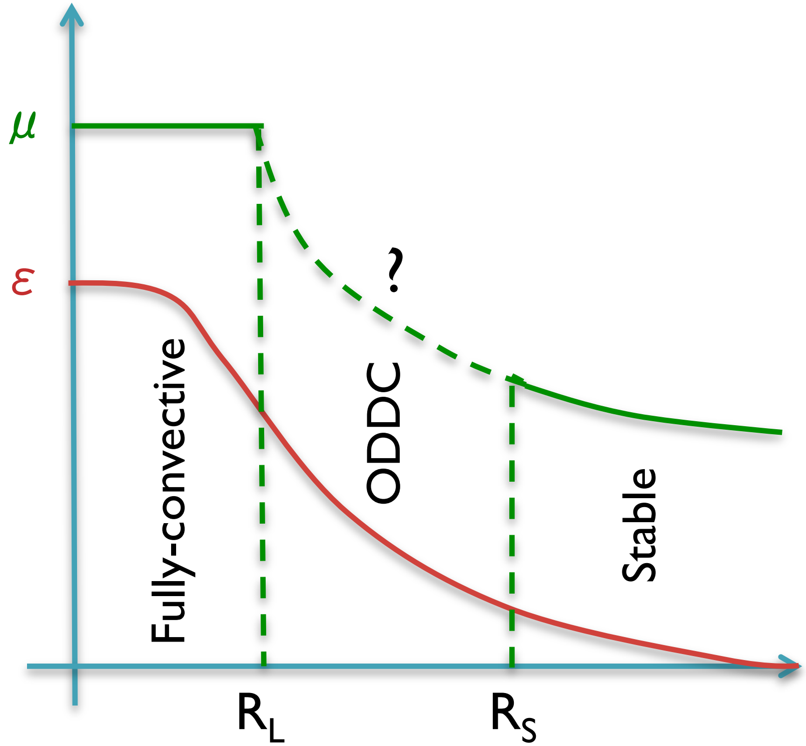
In both cases, mu-gradients build up outside the convective core, but for different reasons.

See reviews by e.g. Spiegel 1969, Langer et al. 1983, Spruit 1992, Merryfield 1995, Shibahashi 2009…



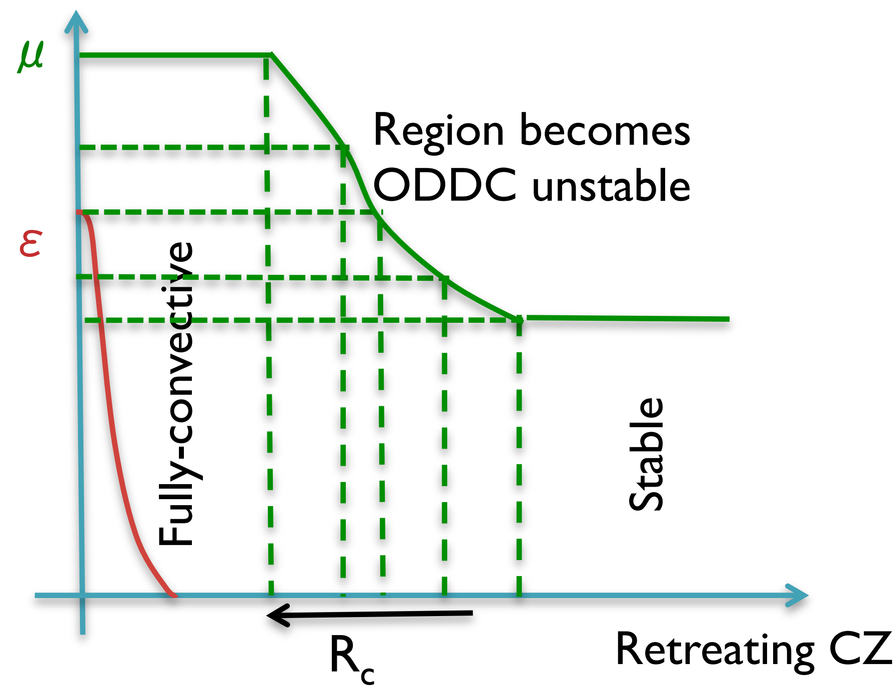
Intermediate-mass stars

- In the 1-2 Msun range, weak dependence of nuclear reactions (pp-chain) on temperature implies He production outside the convective core as estimated using Schwarzschild-radius.
- Ledoux-based radius further out, region in-between is ODDC-unstable.



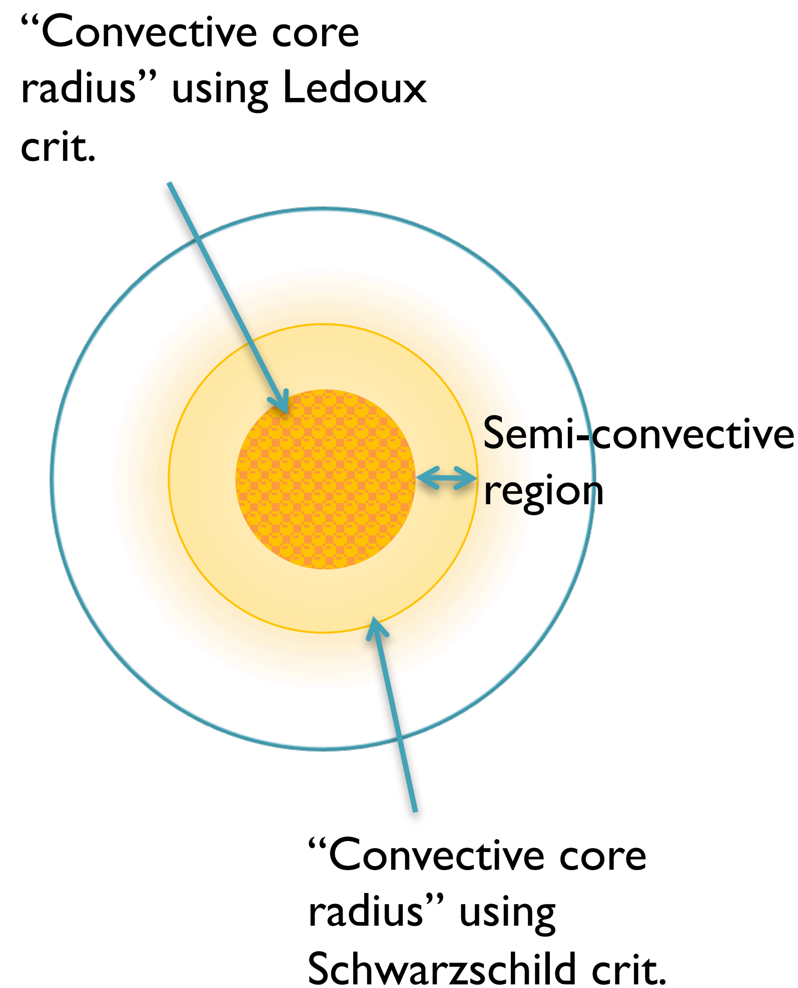
High-mass stars

- In the >10Msun mass range, nuclear reactions strongly concentrated within the convective core
- Opacity in the core ~ 1+X, decreases with X as nuclear reactions proceed. This causes the core to shrink with time as more of the energy can be transported radiatively.
- Shrinking core leaves behind concentric shells of progressively higher He content



Mu-gradients outside convective cores

In all these cases, the main question asked is "How much mixing does semiconvection cause?"



Disclaimer:

The list of relevant DDC-unstable situations presented here is not exhaustive!