A Model of the Effect of Image Motion in the Radon Transform Domain

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***Abstract*— One of the most fundamental properties of the Radon (projection) transform is that shifting of the image results in shifted projections. This useful property relates *translational* motion in the image to simple displacement in the projections. It is far from clear, however, how more general types of motion in the image domain will be manifested in the projections. In this paper, we will present a model for this phenomenon in the general case; namely, we develop a generalization of the shift property of the Radon transform. We study various properties of the apparent projected motion implied by the model, and study the case of affine motion in particular. We also present illustrative examples, and briefly discuss the *inverse* problem implied by the *forward* model developed herein, along with some possible applications**

***Index Terms*— Affine motion, optical flow, projection, Radon transform, shift property.**

I. Motion in the Projection Domain

THE SHIFT property of the Radon transform has found applications in many areas of image processing. For instance, in translational motion estimation from a video sequence [1], [2], and the related problem of image registration [3]. More importantly, projections acquired while the subject undergoes linear motion can be corrected using this property before a reconstruction of the image is attempted.

The shift property of the Radon transform shows that translational motion in the image domain results in translational motion in the projection domain. More specifically, if $g(p, \theta) = \mathcal{R}_\theta[f]$ is the projection of $f(x, y)$ at angle θ defined by

$$g(p, \theta) = \mathcal{R}_\theta[f] = \iint_D f(x, y)\delta(p - x\cos(\theta) - y\sin(\theta))\,dx\,dy \quad (1)$$

we have $\mathcal{R}_\theta[f(x - v_{0x}, y - v_{0y})] = g(p - v_0^T w(\theta), \theta)$, where $v_0 = [v_{0x}, v_{0y}]^T$ and $w(\theta) = [\cos\theta, \sin\theta]^T$ is a unit direction vector.

To the extent that the underlying motion in the image domain can be adequately modeled as translational, this shifting property of the Radon transform is exceedingly useful in applications. More generally, however, one might naturally wonder what happens in the projection domain if the motion in the image domain is *not* a simple displacement. As an example, respiratory motion during CAT scans can be modeled as a combination of expansion (magnification) and displacement [4]. The shifting property of the Radon transform is no longer adequate in describing the effect of general motion in the image on the projections. Hence, a generalization is clearly needed. In this paper, we will discuss such generalizations and study some of their fundamental implications and properties. In particular, we will study the case of affine motion and provide some illustrative examples.

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To begin our development of a model for projected motion, we first state two useful differentiation properties of the Radon transform, which will be invoked later in the paper.

P1—Transform of Derivatives: Let $L(\partial/\partial x, \partial/\partial y)$ denote a linear differential operator, and write the direction vector $w(\theta) = [w_1, w_2]^T$. We have

$$\mathcal{R}_\theta[Lf] = L(w_1\partial/\partial p, w_2\partial/\partial p)g(p, w). \quad (2)$$

In particular, if L is a homogeneous polynomial of degree m with constant coefficients, then

$$\mathcal{R}_\theta[Lf] = L(w)\,\frac{\partial^m g(p, w)}{\partial p^m}. \quad (3)$$

For instance, a useful corollary is

$$\mathcal{R}_\theta[v_0^T \nabla f] = v_0^T w\,\frac{\partial g(p, w)}{\partial p}. \quad (4)$$

P2—Derivatives of the Transform: For integers k and l

$$\frac{\partial^{k+l} g(p, w)}{\partial w_1^k\,\partial w_2^l} = \left(-\frac{\partial}{\partial p}\right)^{k+l} \mathcal{R}_\theta\left[x^k y^l f(x, y)\right] \quad (5)$$

where it must be kept in mind that when derivatives with respect to components of w are computed, the vector w is initially *not* considered a unit vector. The derivatives may later be evaluated for unit direction vectors.

Now, let us consider an image sequence $f(x, y, t)$, which evolves in time according to the spatially varying motion vector field $v(x, y) = [v_1(x, y), v_2(x, y)]^T$. Also, consider its corresponding Radon transform sequence $g(p, \theta, t)$, obtained by computing the Radon transform of f for every fixed t. What we aim to show is that, subject to some conditions, the displaced image $f(x + v_1\Delta t, y + v_2\Delta t, t + \Delta t)$ has a corresponding Radon transform, which we can denote by $g(p + u\Delta t, \theta, t + \Delta t)$, where $u = u(p, \theta, t)$ is the (scalar) motion field induced in the projection domain by motion field v in the image domain. That is, we show that *locally*, the function u exists and is well defined, and that it adequately reflects the behavior of motion induced in the projection domain.

For a sufficiently small time increment Δt, a first-order Taylor series expansion of f is as follows:

$$f(x+v_1\Delta t,\, y+v_2\Delta t,\, t+\Delta t) \approx f(x,\,y,\,t) + v_1\,\frac{\partial f}{\partial x}\,\Delta t + v_2\,\frac{\partial f}{\partial y}\,\Delta t + \frac{\partial f}{\partial t}\,\Delta t \tag{6}$$

$$= f(x,\,y,\,t) + v^T\nabla f\Delta t + \frac{\partial f}{\partial t}\,\Delta t. \tag{7}$$

Next, we consider the Radon transform applied to both sides of the above:

$$\mathcal{R}_\theta[f(x+v_1\Delta t,\, y+v_2\Delta t,\, t+\Delta t)] \approx \mathcal{R}_\theta\left[f(x,\,y,\,t) + v^T\nabla f\Delta t + \frac{\partial f}{\partial t}\,\Delta t\right] \tag{8}$$

$$= g(p,\,\theta,\,t) + \mathcal{R}_\theta\left[v^T\nabla f\right]\Delta t + \frac{\partial g(p,\,\theta,\,t)}{\partial t}\,\Delta t. \tag{9}$$

Now *define* the function $u(p,\,\theta,\,t)$ (henceforth referred to as the *projected motion* by)

$$u(p,\,\theta,\,t) = \frac{\mathcal{R}_\theta\left[v^T\nabla f(x,y,t)\right]}{\partial g(p,\,\theta,\,t)/\partial p}. \tag{10}$$

Clearly, this function is well-defined only when $\partial g(p,\,\theta,\,t)/\partial p \neq 0$, and when $f(x,\,y,\,t)$ is differentiable. We will discuss these requirements in more depth a bit later. For now, assuming that u is thus well-defined, if we replace its definition into (9), we have

$$\mathcal{R}_\theta[f(x+v_1\Delta t,\, y+v_2\Delta t,\, t+\Delta t)] \approx g(p,\,\theta,\,t) + u(p,\,\theta,\,t)\,\frac{\partial g(p,\,\theta,\,t)}{\partial p}\,\Delta t + \frac{\partial g(p,\,\theta,\,t)}{\partial t}\,\Delta t. \tag{11}$$

The right-hand side of (11) now appears quite similar to a Taylor series expansion of $g(p,\,\theta,\,t)$. In fact, if $u(p,\,\theta,\,t)$ can be replaced by dp/dt, we will have exactly the first-order Taylor series of g on the right-hand side. We can make this substitution only when the differential equation

$$\frac{dp}{dt} = u(p,\,\theta,\,t) \tag{12}$$

has a solution, for any fixed θ, over the support of g. The existence and uniqueness theorem for first-order ordinary differential equations [5] states that a unique solution to (12) will exist when $u(p,\,\theta,\,t)$ is continuously differentiable (or C^1); that is, $\partial u/\partial p$ must exist and be continuous[1] on a compact subset of the p-axis. Referring to the definition of u in (10), we can see that if we require that the vector field v be C^1 and that f be C^2, then $\partial u/\partial p$ exists, it is continuous, and is given by

$$\frac{\partial u}{\partial p} = \frac{\left(\partial\mathcal{R}_\theta[v^T\nabla f]/\partial p\right)(\partial g/\partial p) - (\partial^2 g/\partial p^2)\mathcal{R}_\theta[v^T\nabla f]}{(\partial g/\partial p)^2} \tag{13}$$

$$= \frac{\mathcal{R}_\theta[w^T\nabla(v^T\nabla f)]\mathcal{R}_\theta[w^T\nabla f] - \mathcal{R}_\theta[\nabla^2 f]\mathcal{R}_\theta[v^T\nabla f]}{(\mathcal{R}_\theta[w^T\nabla f])^2} \tag{14}$$

where $\nabla^2 f$ denotes the Laplacian of f, and the last identity follows by invoking the differentiation properties described earlier. Note that, as before, we have assumed that $\partial g/\partial p \neq 0$. We note here that, in practice, where u is to be computed from the projections alone, (12) is the relevant equation. That is, u is considered a spatial transformation (or warping) over time in the projection domain.

Taking both f and v to be defined over the same compact region of the plane (the image region), the following proposition, which is the main result of this paper, follows directly from the above definitions and arguments.

Proposition 1—Projected Motion: Consider the image sequence $f(x,\,y,\,t)$, assumed to be twice continuously differentiable (or C^2), which evolves according to the C^1 vector field $v(x,\,y)$. Then, for any $(p,\,\theta,\,t)$, for which $\partial g/\partial p \neq 0$, there exists a C^1 function $u(p,\,\theta,\,t)$ such that, to first order

$$\mathcal{R}_\theta[f(x+v_1\Delta t,\, y+v_2\Delta t,\, t+\Delta t)] \approx g(p+u\Delta t,\, \theta,\, t+\Delta t) \tag{15}$$

for sufficiently small Δt. Furthermore, the function u is given by the identity

$$u(p,\,\theta,\,t)\,\frac{\partial g(p,\,\theta,\,t)}{\partial p} = \mathcal{R}_\theta\left[v^T\nabla f(x,\,y,\,t)\right]. \tag{16}$$

We term this relationship the *differential* **projected motion identity** (PMI). □

A straightforward corollary of the above result is that under the same assumptions, we have

$$\frac{dg}{dt} = \mathcal{R}_\theta\left[\frac{df}{dt}\right]. \tag{17}$$

That is, locally, the projection of the total derivative of f is the total derivative of the projection of f ($\mathcal{R}_\theta$ and the total derivative operation commute). An immediate consequence is that if the optical flow brightness constraint $df/dt = 0$ is assumed to hold in the image domain, then (17) implies that this constraint also holds in the projection domain: $dg/dt = 0$, with motion in this domain given by (16).

The PMI is a natural generalization of the shift property of the Radon transform and is reduced to the standard shift property if the motion vector is spatially invariant. In particular, if the motion vector $v = v_0$ is spatially invariant, then property P1 gives

$$\mathcal{R}_\theta\left[v_0^T\nabla f\right] = v_0^T w(\theta)\,\frac{\partial g}{\partial p} \tag{18}$$

which, when compared to (16), yields $u = v_0^T w(\theta)$, as expected. Furthermore, it is worth noting that as with the shift property, the PMI holds in any dimension. That is, if the Radon transform of a scalar function of n real variables is defined as its integrals over hyperplanes of dimension $n-1$, the arguments presented above would yield the same result except that v would be an n-dimensional vector field.

[1] This will imply that u and $\partial u/\partial p$ are also bounded on the same interval.

II. Properties of the Projected Motion

Several interesting properties and implication of the projected motion, and the model in (16) are worth considering. First, we note that u may be time-varying even if the vector field v is not. This is due to the dependence of u on the gradient of the image, which varies with time. Another observation worth making is that by invoking the directional derivative property P1, we can rewrite $\partial g/\partial p$ in the image domain and express the PMI as follows:

$$u(p, \theta, t)\mathcal{R}_\theta\left[w^T\nabla f\right] = \mathcal{R}_\theta\left[v^T\nabla f\right]. \quad (19)$$

The insight we gain here is that u is expressible as the *ratio* of two projections; namely, the projection of the directional derivative of the image parallel to v (sometimes called the *advective* derivative of f), and the directional derivative of the image parallel to the unit vector $w(\theta)$, when the latter projection is not zero. Intuitively, at points where $\mathcal{R}_\theta[w^T\nabla f]$ vanishes, there is no perceived motion in the projection taken at angle θ, and hence, as expected, u is not well defined. It is also interesting to note that in each direction of projection, the correspondence between the vector field v and the function u is not unique. Namely, for a given θ, both v and $v + v_\perp$ yield the same u if $v_\perp$ is such that $\mathcal{R}_\theta[v_\perp^T\nabla f] = 0$.

It is important to note that another (less general) form of the PMI, based upon more restrictive global conservation assumptions, is also possible. Namely, Fitzpatrick [6] considered f and v both C^1, where f is to represent the density of some *conserved* quantity. That is

$$\frac{\partial f}{\partial t} + \text{div}(fv) = 0 \quad (20)$$

which is the familiar continuity equation of fluid dynamics. Write

$$\frac{\partial f}{\partial t} + \text{div}(fv) = \frac{\partial f}{\partial t} + \frac{\partial(fv_1)}{\partial x} + \frac{\partial(fv_2)}{\partial y} = 0. \quad (21)$$

Taking the Radon transform of both sides of (21), and applying property P1, we have[2]

$$\begin{aligned}\frac{\partial g}{\partial t} &+ \mathcal{R}_\theta\left[\frac{\partial(fv_1)}{\partial x}\right] + \mathcal{R}_\theta\left[\frac{\partial(fv_2)}{\partial y}\right] \\ &= \frac{\partial g}{\partial t} + w_1\frac{\partial}{\partial p}\mathcal{R}_\theta[fv_1] + w_2\frac{\partial}{\partial p}\mathcal{R}_\theta[fv_2] \quad (22)\\ &= \frac{\partial g}{\partial t} + \frac{\partial}{\partial p}\mathcal{R}_\theta\left[fv^Tw\right] = 0. \quad (23)\end{aligned}$$

Now if we define

$$u_c(p, \theta, t)g(p, \theta, t) = \mathcal{R}_\theta\left[fv^Tw\right] \quad (24)$$

whenever $g \neq 0$, and replace this definition into (23), we obtain a continuity equation for g:

$$\frac{\partial g}{\partial t} + \frac{\partial(u_c g)}{\partial p} = 0. \quad (25)$$

The identity (24) is the PMI implied by the conservation assumption (hence, the subscript c on u). Similar to (16), (24) also implies a description of u_c as the *ratio* of two projection: $u_c\mathcal{R}_\theta[f] = \mathcal{R}_\theta[fv^Tw]$. That is, u_c is the ratio of the projection of the *flux* (fv) in the direction of w, to the projection of f itself in the same direction. The issue of whether (16) or (24) should be used in describing the nature of motion in the projection domain is a matter of which assumptions are most adequate in describing the application at hand. However, while the differential form of PMI makes slightly stronger smoothness assumptions on f, it is more generally applicable as it is not based on a global conservation assumption. We note that naturally, as with any differential model of motion, the motion in the projections is not well-defined in (16) when the local gradient $\partial g/\partial p$ is null. If (16) is to be used as a means of *measuring* u from *image data*, then additional assumptions such as smoothness may have to be invoked to compute u when the gradient is near zero. Alternatively, other models of the projected motion such as the one described above can be invoked, if the underlying assumptions are appropriate. We distinguish the two models of projected motion by referring to (24) as the *integral* (or conservative) PMI, whereas without this qualification, we understand PMI to mean the *differential* version in (16).

A number of interesting properties of projected motion can be derived directly from the properties of the Radon transform stated earlier and in [7]. For instance, it follows from the linearity of the Radon transform that for a given image f, if u and u' are the projected motions resulting from the vector fields v and v', respectively, then the projected motion field resulting from $av + bv'$ is simply $au + bu'$, where a and b are arbitrary scalars. This, in turn, implies that if a given vector field v is decomposed according to Helmholtz's theorem [8] into its irrotational and solenoidal components as $v = v_I + v_S$, the projected motion field u has a decomposition of the same kind: $u = u_I + u_S$. Other useful properties[3] of u include periodicity: $u(p, \theta + 2k\pi, t) = u(p, \theta, t)$, and *anti*-symmetry: $u(p, \theta, t) = -u(-p, \theta + \pi, t)$. Finally, it is well known [7], [9] that the moments of the projections are linearly related to the moments of the image. Of particular interest is the case of zeroth-order moments of a function and its Radon transform, which are, in fact, equal. That is, if f is thought of as a density, then the total mass given by the integral of f over its domain of definition is equal to the total area under any projection in an arbitrary direction. Applying this result to the *differential* PMI, we obtain

$$\iint v^T\nabla f(x, y, t)\,dx\,dy = \int u(p, \theta, t)\frac{\partial g}{\partial p}\,dp \quad (26)$$

which states the intuitively pleasing result that projection conserves the average advective derivative of f. Applied to the *integral* PMI, we get

$$\iint f(x, y, t)v^Tw\,dx\,dy = \int u_c(p, \theta, t)g\,dp \quad (27)$$

which means that the total flux in any direction is conserved by projection.

[2] This is a generalized rederivation of Fitzpatrick's result in [6].

[3] Linearity, periodicity, and antisymmetry properties are also satisfied by u_c.

III. Analysis of Affine Motion in the Projection Domain

Any motion field can be locally approximated (to first order) by affine motion. Hence, it is important to consider the class of motions given by

$$v = v_0 + M\begin{bmatrix} x \\ y \end{bmatrix}, \qquad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{28}$$

where v_0 is a fixed vector denoting translational motion.

To see specifically how affine transformation behaves in the projection domain, let us consider warping an image $f(x, y)$ by such a transformation. Letting $f(x, y) = f(x, y, 0)$, if we compute the derivative of both sides of the differential PMI with respect to p and invoke P1 and the linearity property of the Radon transform, we get

$$\begin{aligned} \frac{\partial}{\partial p}\left(u\,\frac{\partial g}{\partial p}\right) &= \frac{\partial}{\partial p}\left(\mathcal{R}_\theta\left[(ax+by)\,\frac{\partial f}{\partial x} + (cx+dy)\,\frac{\partial f}{\partial y}\right] + \mathcal{R}_\theta\left[v_0^T \nabla f\right]\right) \qquad (29)\\ &= \frac{\partial}{\partial p}\,\mathcal{R}_\theta\left[x\left(a\,\frac{\partial f}{\partial x} + c\,\frac{\partial f}{\partial y}\right)\right] + \frac{\partial}{\partial p}\,\mathcal{R}_\theta\left[y\left(b\,\frac{\partial f}{\partial x} + d\,\frac{\partial f}{\partial y}\right)\right] + \frac{\partial}{\partial p}\left(v_0^T w\,\frac{\partial g}{\partial p}\right). \qquad (30)\end{aligned}$$

Writing the direction vector $w = [w_1, w_2]^T$ and using property P2, we can rewrite (30) as follows:

$$\begin{aligned} \frac{\partial}{\partial p}\left(u\,\frac{\partial g}{\partial p}\right) &= -\frac{\partial}{w_1}\,\mathcal{R}_\theta\left[a\,\frac{\partial f}{\partial x} + c\,\frac{\partial f}{\partial y}\right] - \frac{\partial}{w_2}\,\mathcal{R}_\theta\left[b\,\frac{\partial f}{\partial x} + d\,\frac{\partial f}{\partial y}\right] + \frac{\partial}{\partial p}\left(v_0^T w\,\frac{\partial g}{\partial p}\right) \qquad (31)\\ &= -\frac{\partial}{w_1}\left[(aw_1+cw_2)\,\frac{\partial g}{\partial p}\right] - \frac{\partial}{w_2}\left[(bw_1+dw_2)\,\frac{\partial g}{\partial p}\right] + \frac{\partial}{\partial p}\left(v_0^T w\,\frac{\partial g}{\partial p}\right) \qquad (32)\\ &= -\frac{\partial}{\partial p}\left[\frac{\partial}{w_1}\,(aw_1+cw_2)g + \frac{\partial}{w_2}(bw_1+dw_2)g - v_0^T w\,\frac{\partial g}{\partial p}\right] \qquad (33)\\ &= -\frac{\partial}{\partial p}\left(\mathrm{tr}(M)g + w^T M \begin{bmatrix} \partial g/\partial w_1 \\ \partial g/\partial w_2 \end{bmatrix} - v_0^T w\,\frac{\partial g}{\partial p}\right). \qquad (34)\end{aligned}$$

Integrating both sides of (34) with respect to p we get[4]

$$\left(u - v_0^T w\right)\frac{\partial g}{\partial p} + \mathrm{tr}(M)g + w^T M \begin{bmatrix} \partial g/\partial w_1 \\ \partial g/\partial w_2 \end{bmatrix}\Bigg|_{|w|=1} = 0. \tag{35}$$

Much can be learned about the general structure of affine motion in the projection domain by considering the representation of images using Hermite polynomials. In particular, consider

$$f(x, y, 0) = \sum_{k,\,l} f_{kl} H_k(x) H_l(y) e^{-x^2-y^2} \tag{36}$$

where $\{H_k(x)H_l(y);\ k, l = 0, 1, 2, \cdots\}$ is the orthogonal basis[5] of Hermite polynomials.

It can be shown (see [10] for details) that for this choice of f, and for sufficiently large p:

$$\begin{bmatrix} \partial g/\partial w_1 \\ \partial g/\partial w_2 \end{bmatrix}\Bigg|_{|w|=1} \approx 2p^2 g w \quad \text{and} \quad \partial g/\partial p \approx -2pg. \tag{37}$$

Substituting these approximations into (35) and solving for u we obtain the following neat asymptotic expression for u:

$$u \approx v_0^T w + \left(w^T M w\right)p. \tag{38}$$

Expanding the quadratic form in M, we have

$$u(p, \theta, 0) \approx v_{0x}\,\cos(\theta) + v_{0y}\,\sin(\theta) + \left(\frac{a+d}{2} + \frac{a-d}{2}\,\cos(2\theta) + \frac{b+c}{2}\,\sin(2\theta)\right)p, \tag{39}$$

in which, interestingly, no term corresponding to pure rotation (i.e., the curl strength $c-b$) appears. In fact, what occurs is that the influence of any pure rotation in the image plane decays essentially as $1/p$ or faster away from the vertex of rotation in the projection domain, and is therefore not reflected in the above asymptotic formula.

IV. Some Examples of Projected Motion

In this section we present two examples. In the first, algebraic expressions for $u(p, \theta, t)$ for an analytic image sequence and vector field are derived. In the second example, we apply the motion model developed here to an image sequence and verify that the resulting estimates of the motion in the projection domain are consistent with our proposed model and our intuitive expectations.

A. Example 1

Let $f(x, y, t) = \exp(-(x - v_1 t)^2 - (y - v_2 t)^2)$, and $v(x, y) = [x, y]^T$. Computing the gradient of f, we have

$$\nabla f = -2(1-t)^2\,\exp\left(-(1-t)^2(x^2+y^2)\right)[x, y]^T \tag{40}$$

which yields (see [7])

$$\begin{aligned} \mathcal{R}_\theta\left[v^T \nabla f\right] &= \mathcal{R}_\theta\left[-2(x^2+y^2)(1-t)^2 \cdot \exp\left(-(1-t)^2(x^2+y^2)\right)\right] \qquad (41)\\ &= \frac{-\sqrt{\pi}\,\exp\left(-p^2(1-t)^2\right)}{|1-t|}\,\left(2p^2(1-t)^2 + 1\right) \qquad (42)\end{aligned}$$

[4]The indeterminate constant resulting from indefinite integration is easily shown to be zero by letting $f = 0$.

[5]While a shortcoming of this representation is that the basis functions $H_k(x)H_l(y)$ are not compactly supported when real images are, the inclusion of the exponential factor makes this representation somewhat more realistic for image processing.

and

$$\frac{\partial g(p,\,\theta,\,t)}{\partial p} = \frac{\partial}{\partial p}\,\mathcal{R}_\theta[f] = \frac{\partial}{\partial p}\,\frac{\sqrt{\pi}\,\exp\bigl(-p^2(1-t)^2\bigr)}{|1-t|} \tag{43}$$

$$= \frac{-2\sqrt{\pi}p(1-t)^2\,\exp\bigl(-p^2(1-t)^2\bigr)}{|1-t|}. \tag{44}$$

Invoking the differential PMI, and solving for u, we obtain

$$u(p,\,\theta,\,t) = p + \frac{1}{2p(1-t)^2} \tag{45}$$

which has (removable) singularities at $p = 0$ and $t = 1$ where $\partial g/\partial p$ vanishes. For this example, the singularities can be explained rather easily. For instance, at $t = 1$, $f(x,\,y,\,1)$ has null gradient and hence there is no perceived motion at all. The explanation for the singularity at $p = 0$ is a bit more subtle. Namely, for $p = 0$, and regardless of the angle, all pixels in the image are moving in a perpendicular direction to $w(\theta)$. Hence, no motion can be measured in the projections.

More generally, considering affine motion as in (28); skipping the details of the computation, we obtain the *exact* expression

$$u(p,\,\theta,\,t) = v_0^T w + \frac{\mathrm{tr}(M) + \bigl(w^T M w\bigr)\bigl(2p^2(1-t)^2 - 1\bigr)}{2p(1-t)^2} \tag{46}$$

which asymptotically agrees with our result in (38).

B. Example 2

In this example, the *diverging trees* image sequence (courtesy of Barron *et al.* [11]) is used to show that the PMI model for motion agrees with actual measurements of motion in the projections. The said image sequence consists of 40 frames, each having 150 × 150 pixels, obtained as the camera moves along its line of sight toward the scene, resulting in a divergent motion field with the focus of expansion located at the center of the image. The twentieth frame, along with some sample motion vectors, are shown in Fig. 1. The exact motion vector field is known,[6] and is well described by $v(x,\,y) \approx 1.1[x,\,y]^T$ (in units of pixels per frame). Projections of the frames were computed in the row and column directions, and from these, using a Fourier transform-based technique described in [12] (which is a refinement of the algorithm in [2]), the motion in the projections was measured. These estimated values are shown as the solid and dashed curves in Fig. 2. The asymptotic model in (38), with $v_0 = [0,\,0]^T$ and $M = 1.1I$, then implies that the predicted motion in the projections at any angle should be $u(p) \approx 1.1p$. These predicted values are displayed in Fig. 2 as circles. It is evident that they generally agree quite well with the directly estimated values while, not surprisingly, the largest errors occur at the center of the plots near the projection of the focus of expansion. Note that the model exhibits a certain degree of robustness to the extent that it is accurate (at least for this simple motion field) even though the images are neither C^2, nor necessarily well represented by the model (36) in terms of Hermite polynomials.

[6] It can be downloaded, along with the image sequence from ftp://csd.uwo.ca/pub/vision/TESTDATA/.

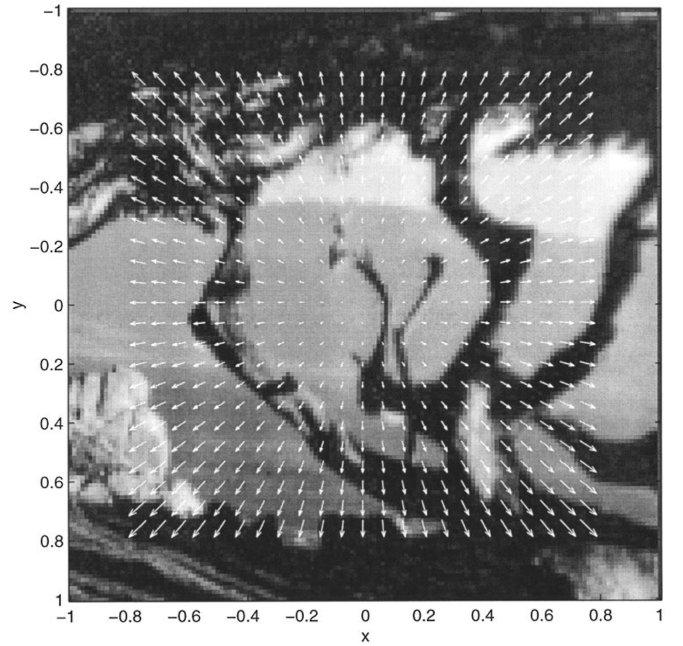

Fig. 1. Frame 20 and the optical flow field for Example 2.

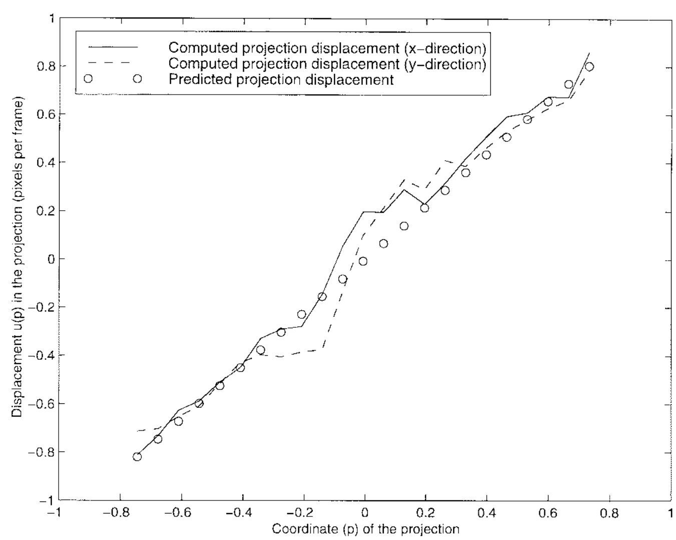

Fig. 2. Measured and predicted motion in the projections for Example 2.

V. Conclusions and Future Directions

We considered the question of modeling the mapping between motion in an image (or image sequence) and its projections. To this end, we developed a local first-order model (the differential projected motion identity) and showed that it produces results that are reasonable and intuitive. We also studied alternative global formulations of the PMI based on conservation assumptions. We derived some basic properties of projected motion, and studied the effect of affine motion in the projection domain using the differential formulation, particularly for a general class of images defined in terms of Hermite polynomials. This analysis revealed that, at least asymptotically, the projected affine motion is itself affine in nature, and that the effect of rotation tends to dissipate as the inverse distance from the vortex in the projection domain, and is hence difficult to measure.

More generally, the PMI can be considered as an indirect measurement equation (or *forward model*) for motion flow in the image domain. This implies an inverse problem. Namely, given measurements of the projections g and hence their respective motion field u (or u_c), how do we reconstruct v? The measurements u_c implied by integral PMI, being inner-product measurements, are transversal in nature [13] and therefore yield information only about the irrotational component of v. The more general case of inverting for v from measurements u implied by differential PMI seems to be a more interesting and challenging inverse problem since these measurements contain some information about both solenoidal and irrotational components of the motion field v. Existing reconstruction algorithms [13]–[15] may be successful in recovering the irrotational part of v from u or u_c. If v is purely solenoidal $(\text{div}(v) = 0)$, however, the continuity equation (20) invoked in the derivation of integral PMI reduces to the familiar optical flow brightness constraint, and u_c conveys *no* information about v in this case.

Generally, questions of existence and uniqueness of solutions, along with numerically well-behaved algorithms for performing the inversion, remain to be studied. This inverse problem has a number of interesting applications. For instance, it has been shown [1], [2] that using (two) projections, we can efficiently estimate translational motion in the image. The natural next step would be to ask whether computationally efficient motion estimation algorithms using projections can be obtained for more general types of motion. As we can see in Section III (38), this appears to be possible in at least the affine case.

A solution to the inverse problem implied by the PMI is useful in any application where it may be difficult or impossible to collect inner product measurements of a vector field. In these cases, it may be possible instead to measure ordinary line integral projections of the density field, compute motion in these projections using existing motion estimation techniques (applied in one dimension), and attempt to invert for the desired higher-dimensional vector field. This is a promising direction of research with many applications which we are currently pursuing. A forthcoming paper will present some of the results of this effort.

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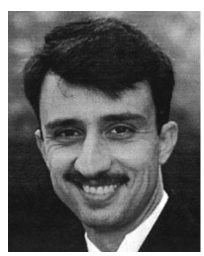

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