# 5.4 Closest Pair of Points

# Closest Pair of Points

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

fast closest pair inspired fast algorithms for these problems

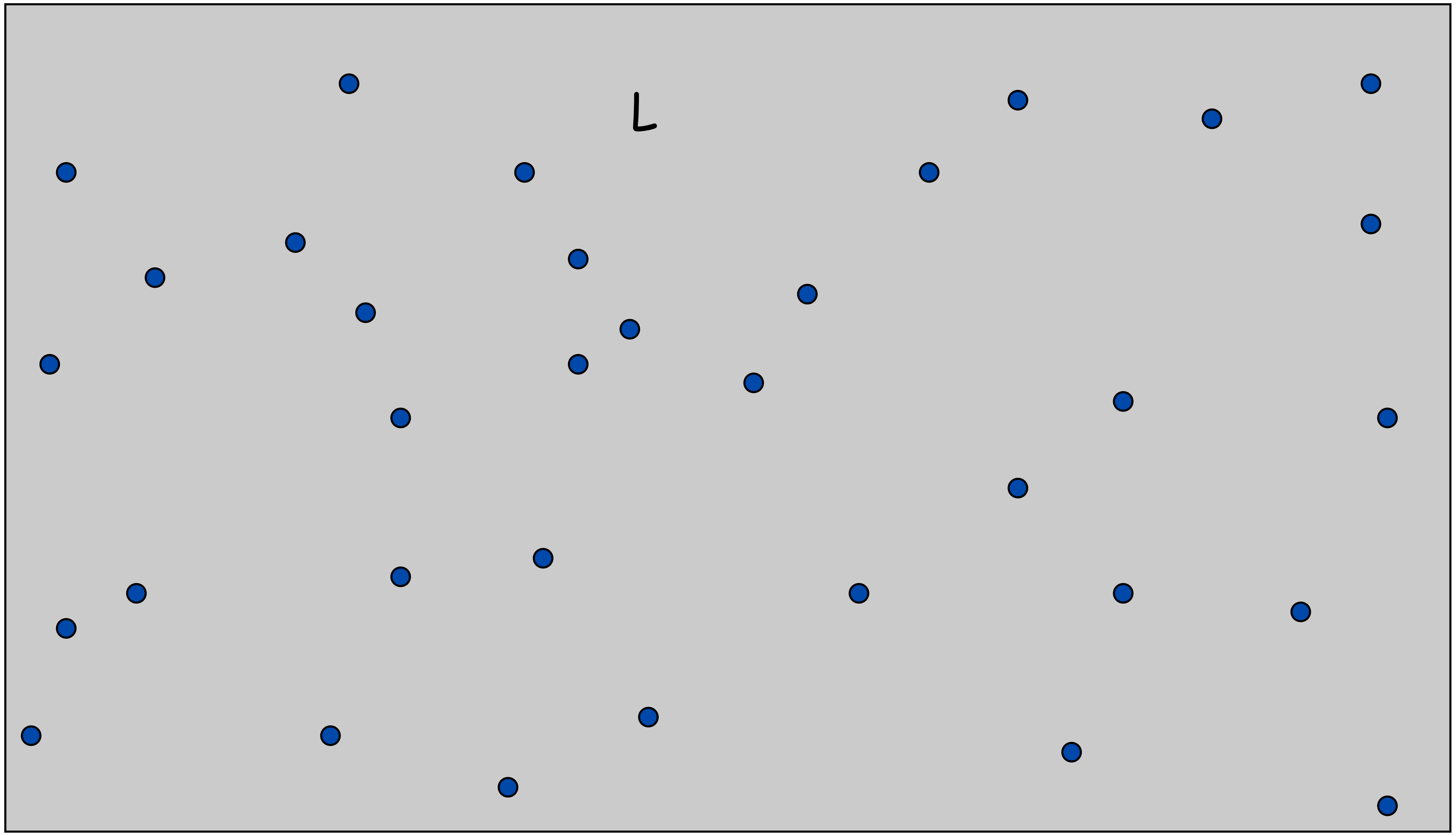
1-D version. $O(n \log n)$ easy if points are on a line.

Assumption. No two points have same x coordinate.

to make presentation cleaner

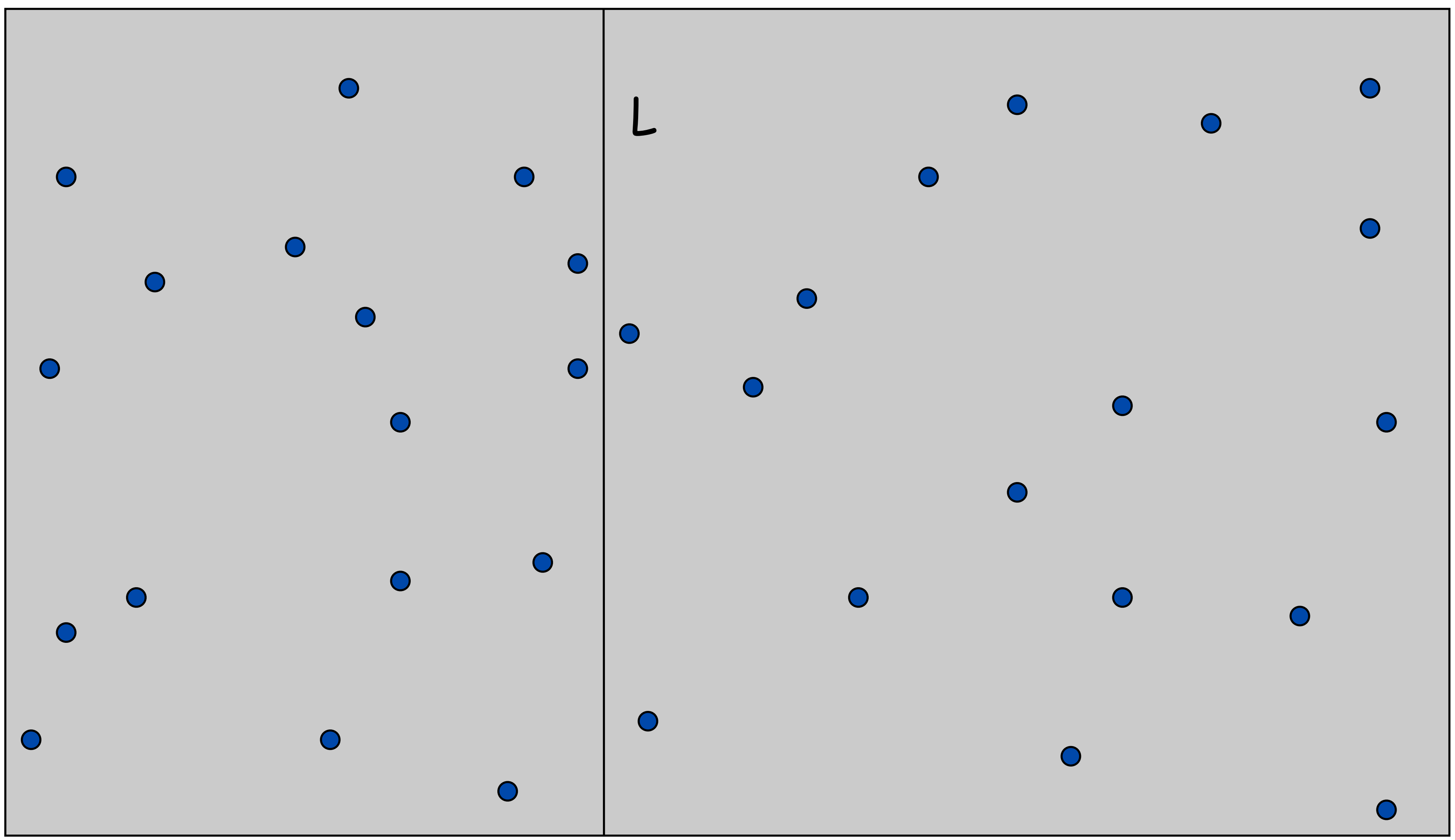
# Closest Pair of Points: First Attempt

Divide. Sub-divide region into 4 quadrants.



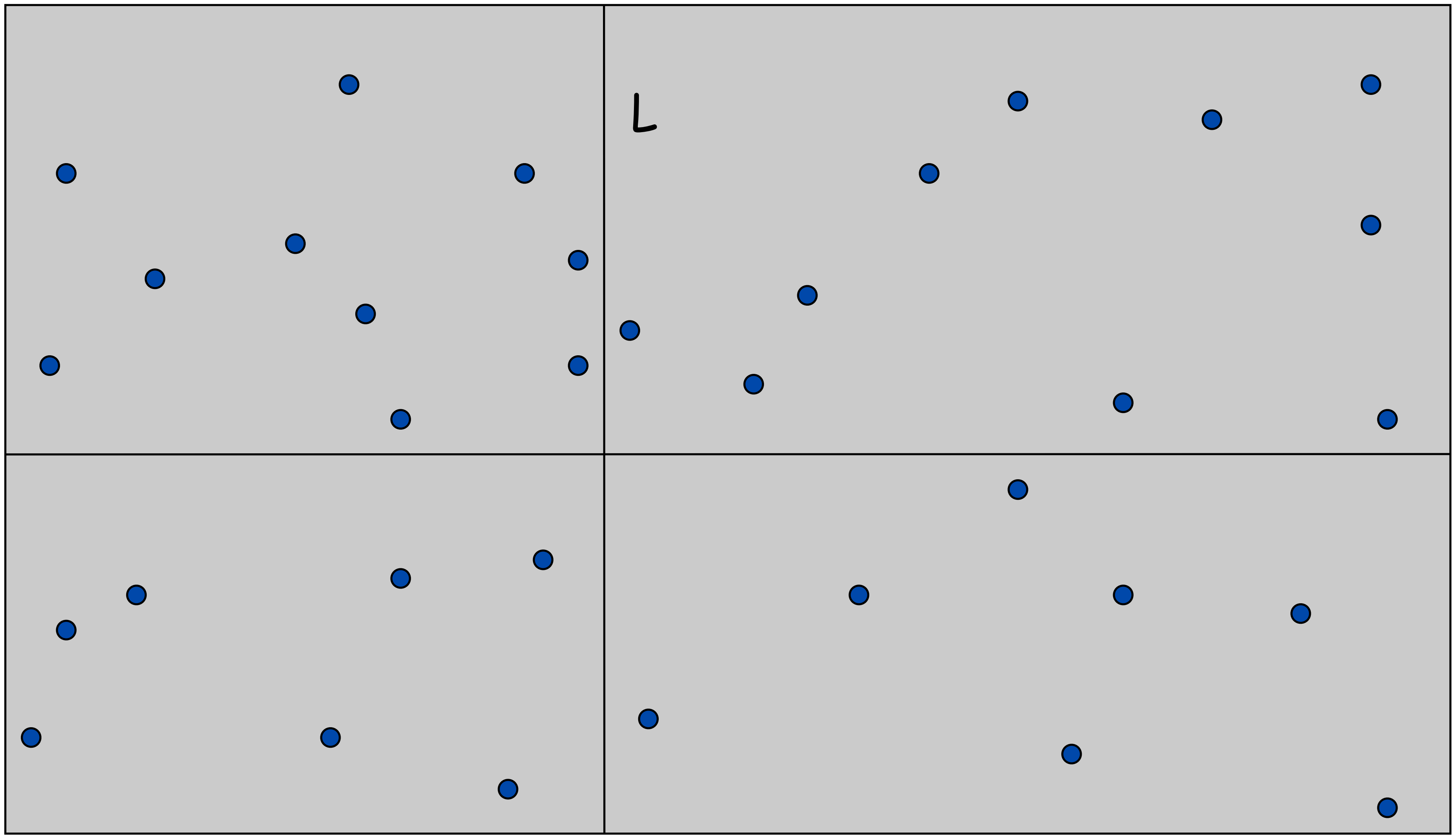
# Closest Pair of Points: First Attempt

Divide. Sub-divide region into 4 quadrants.



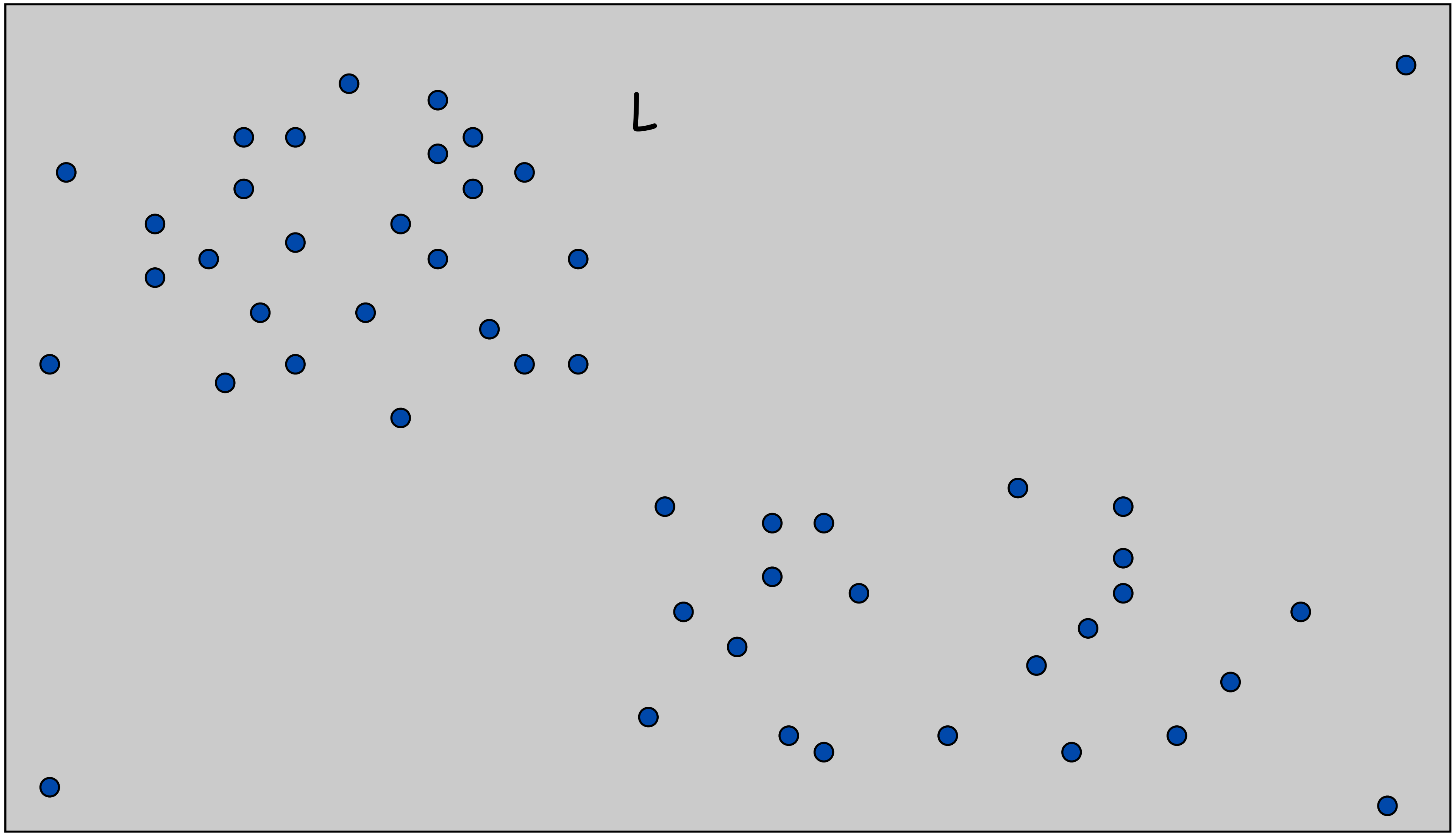
# Closest Pair of Points: First Attempt

Divide. Sub-divide region into 4 quadrants.



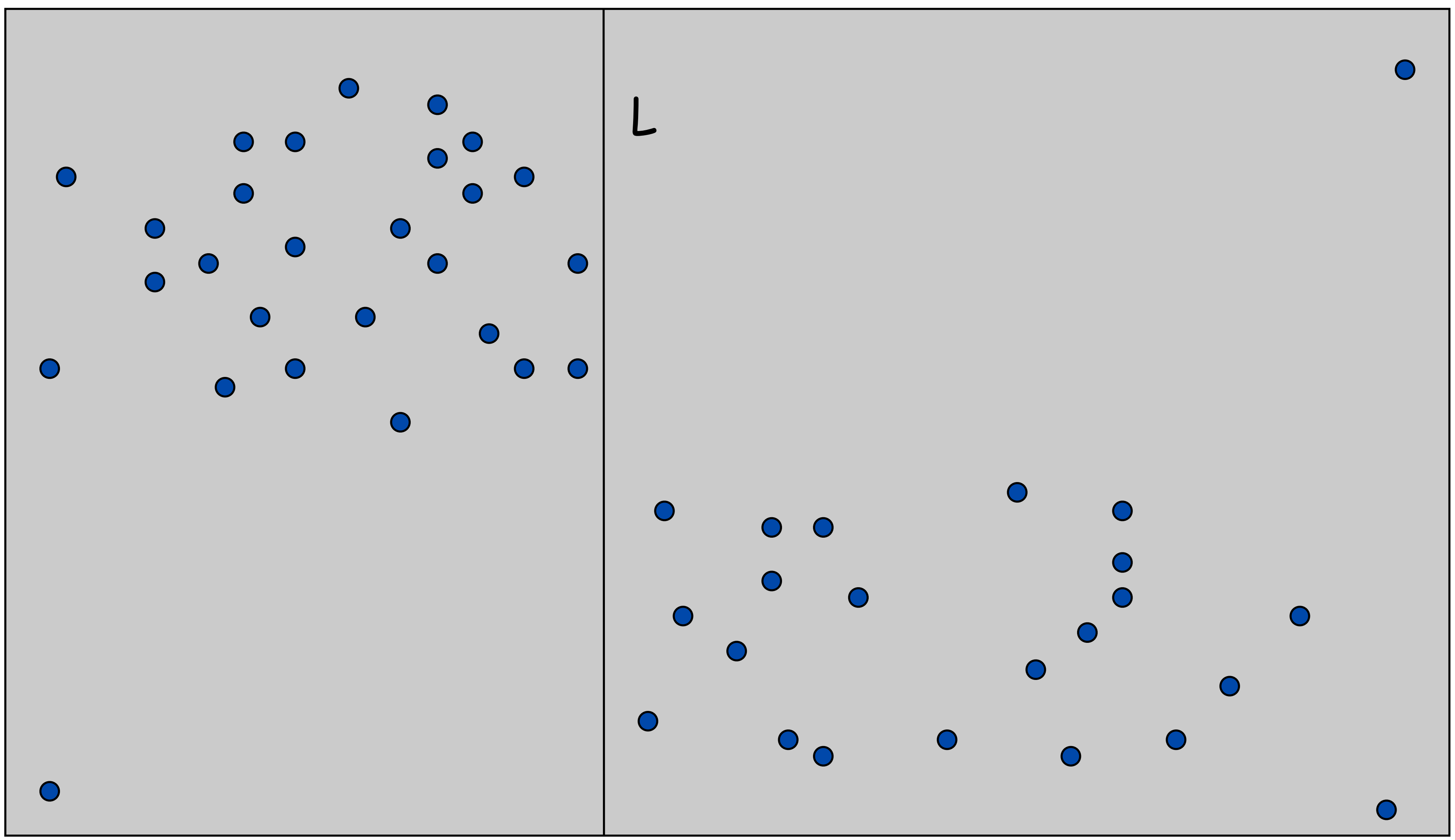
# Closest Pair of Points: First Attempt

Divide. Sub-divide region into 4 quadrants.
Obstacle. Impossible to ensure n/4 points in each piece.



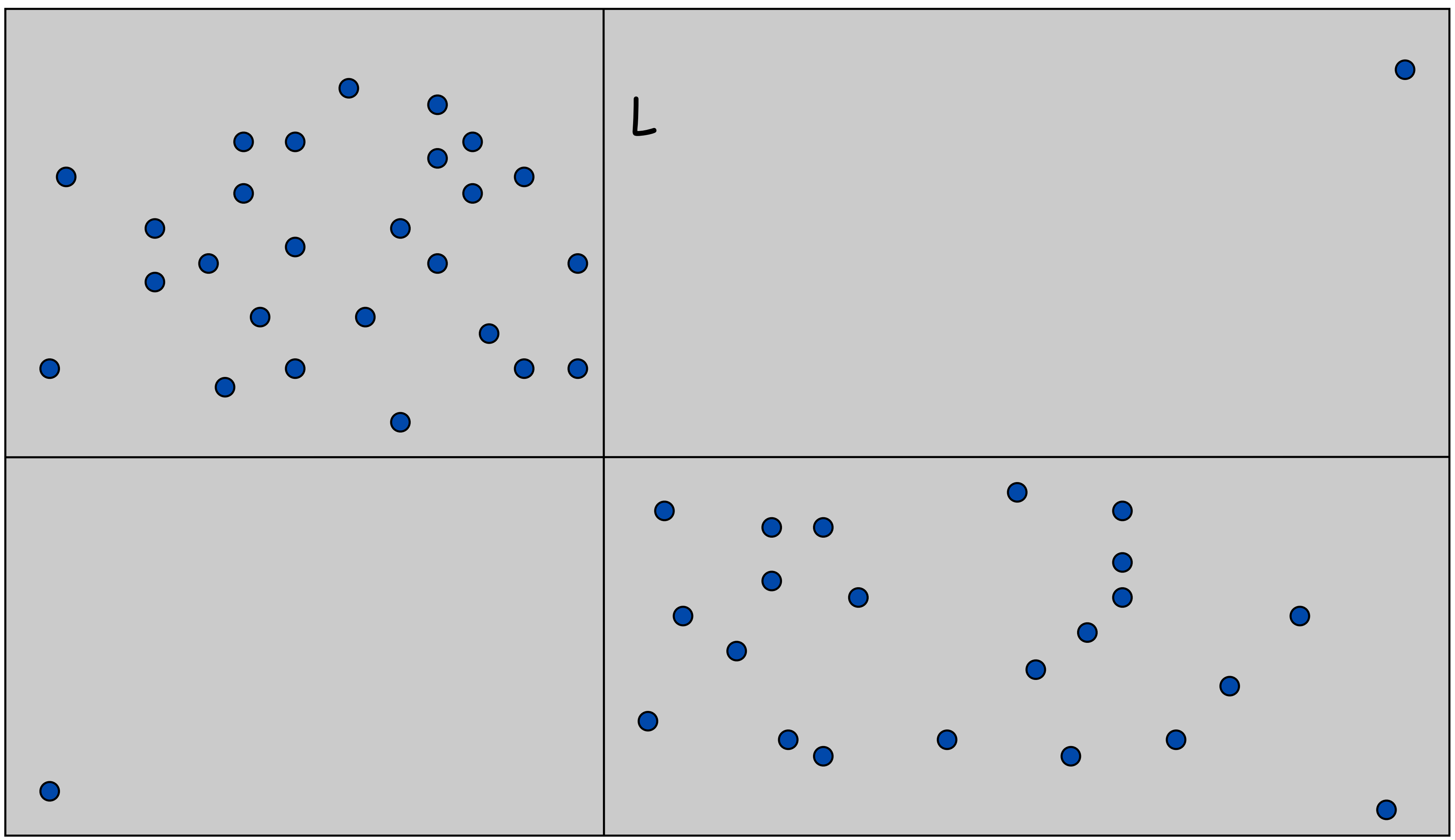
# Closest Pair of Points: First Attempt

Divide. Sub-divide region into 4 quadrants.
Obstacle. Impossible to ensure n/4 points in each piece.



# Closest Pair of Points: First Attempt

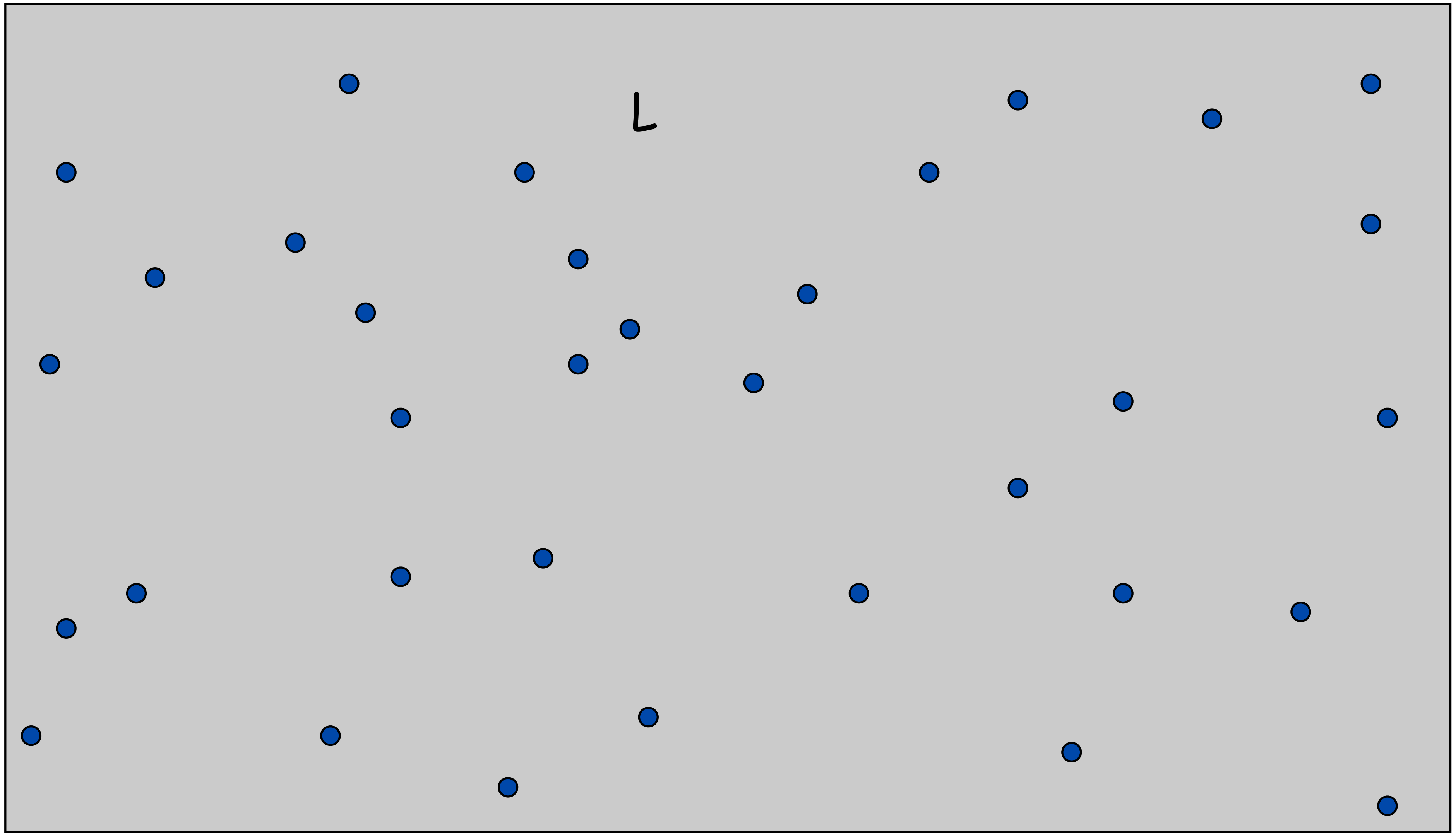
Divide. Sub-divide region into 4 quadrants.
Obstacle. Impossible to ensure n/4 points in each piece.



# Closest Pair of Points

Algorithm.

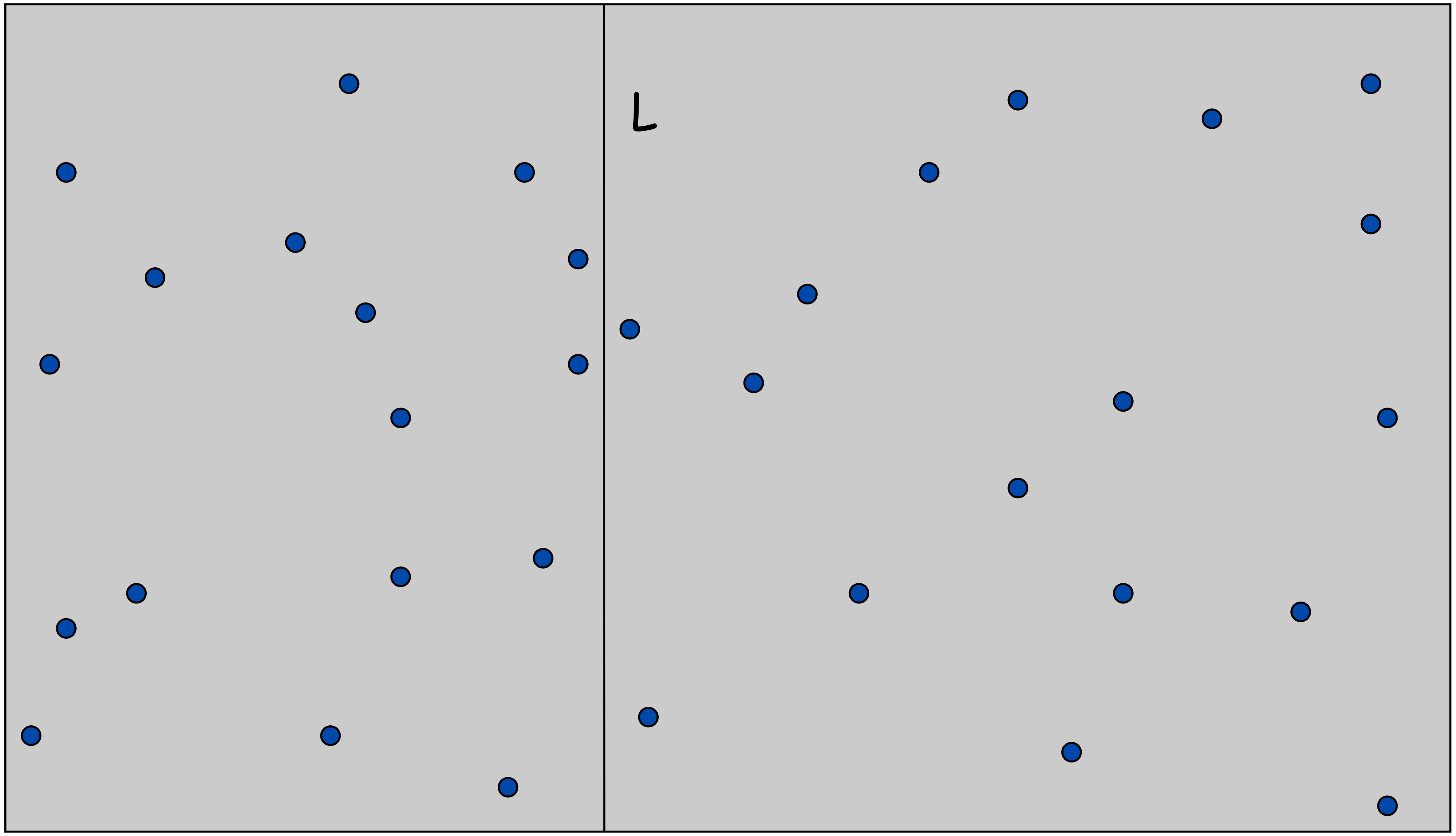
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.



# Closest Pair of Points

Algorithm.

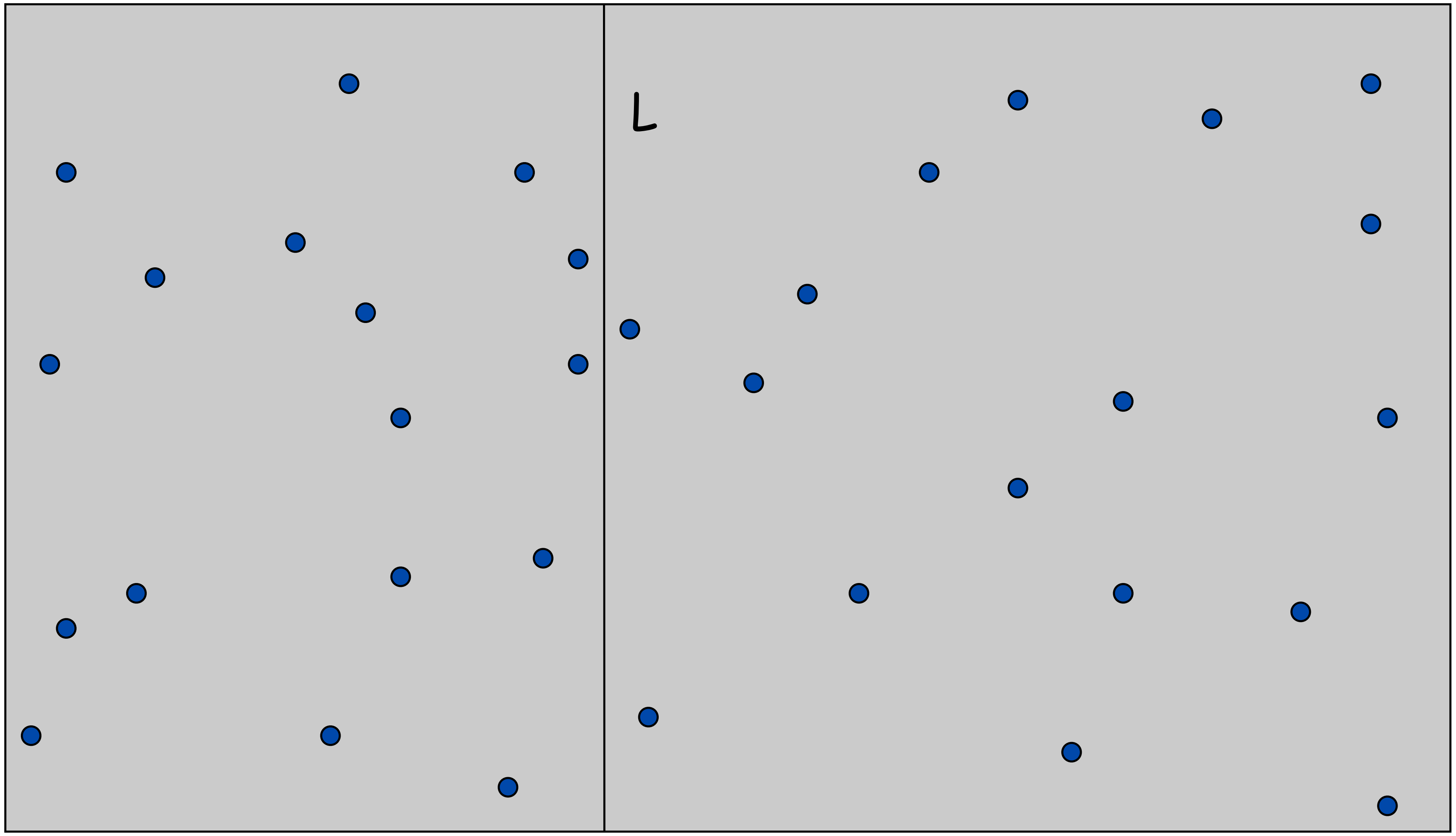
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.



# Closest Pair of Points

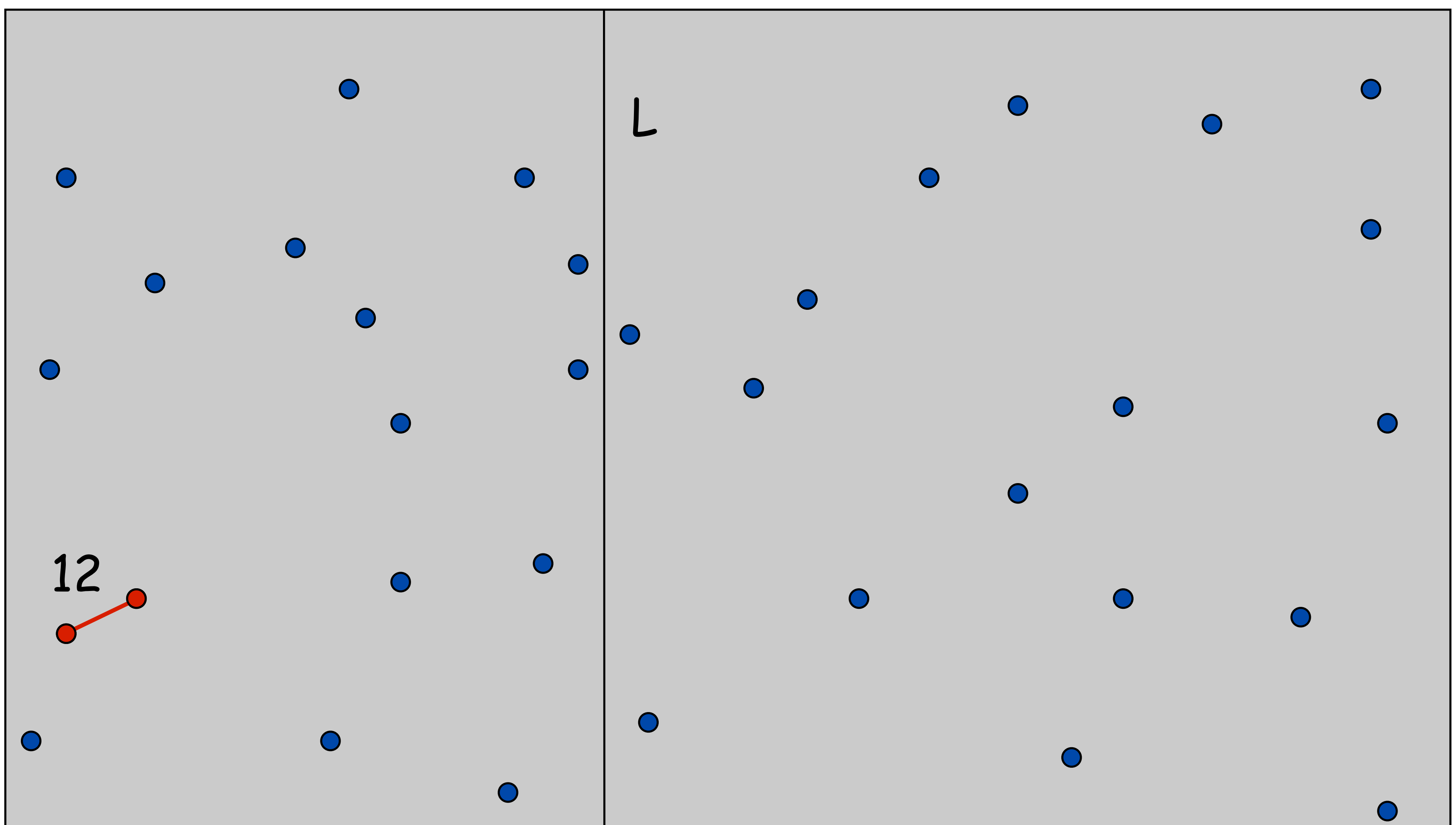
Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.



# Closest Pair of Points

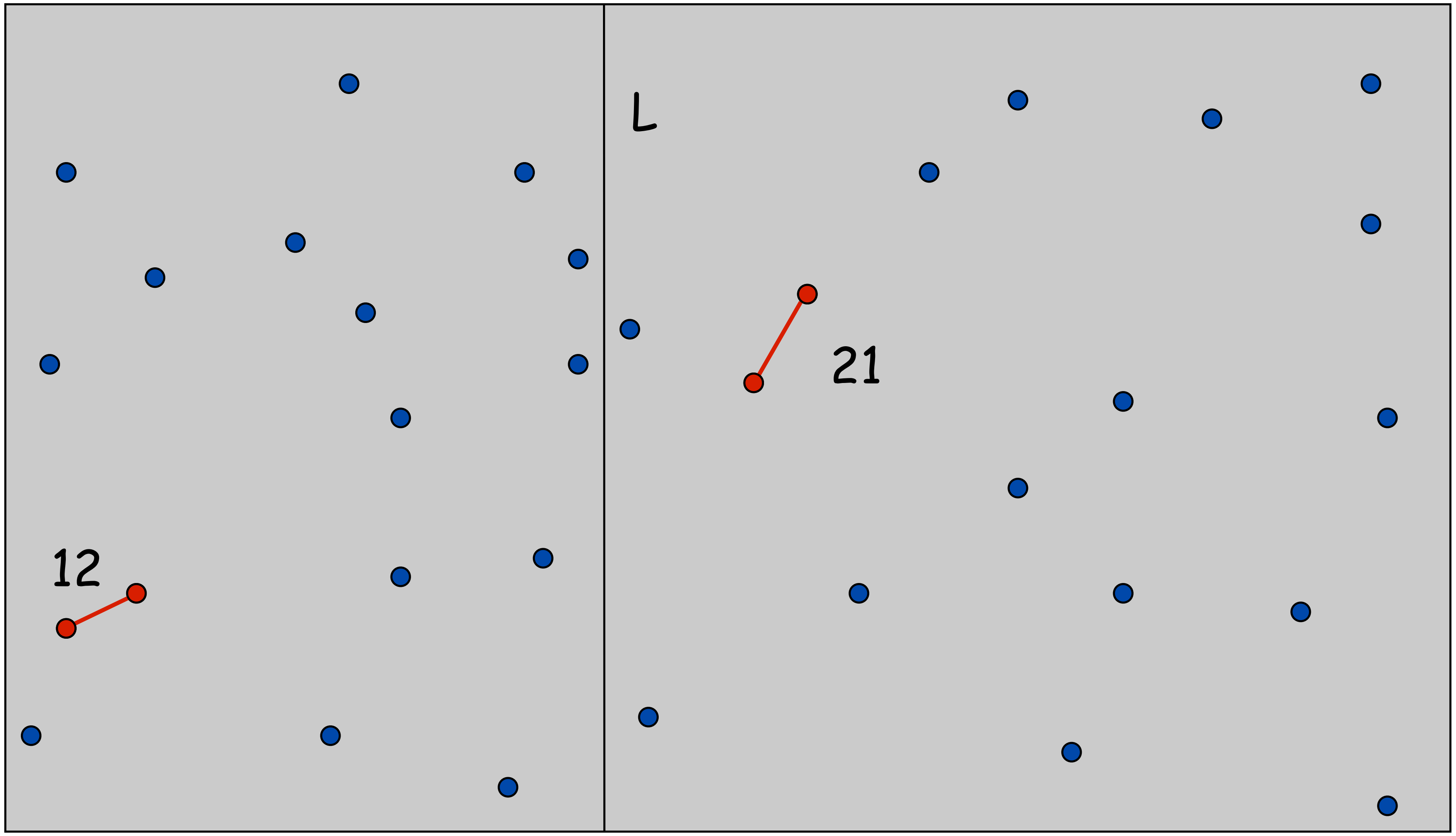
Algorithm.
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.



# Closest Pair of Points

Algorithm.

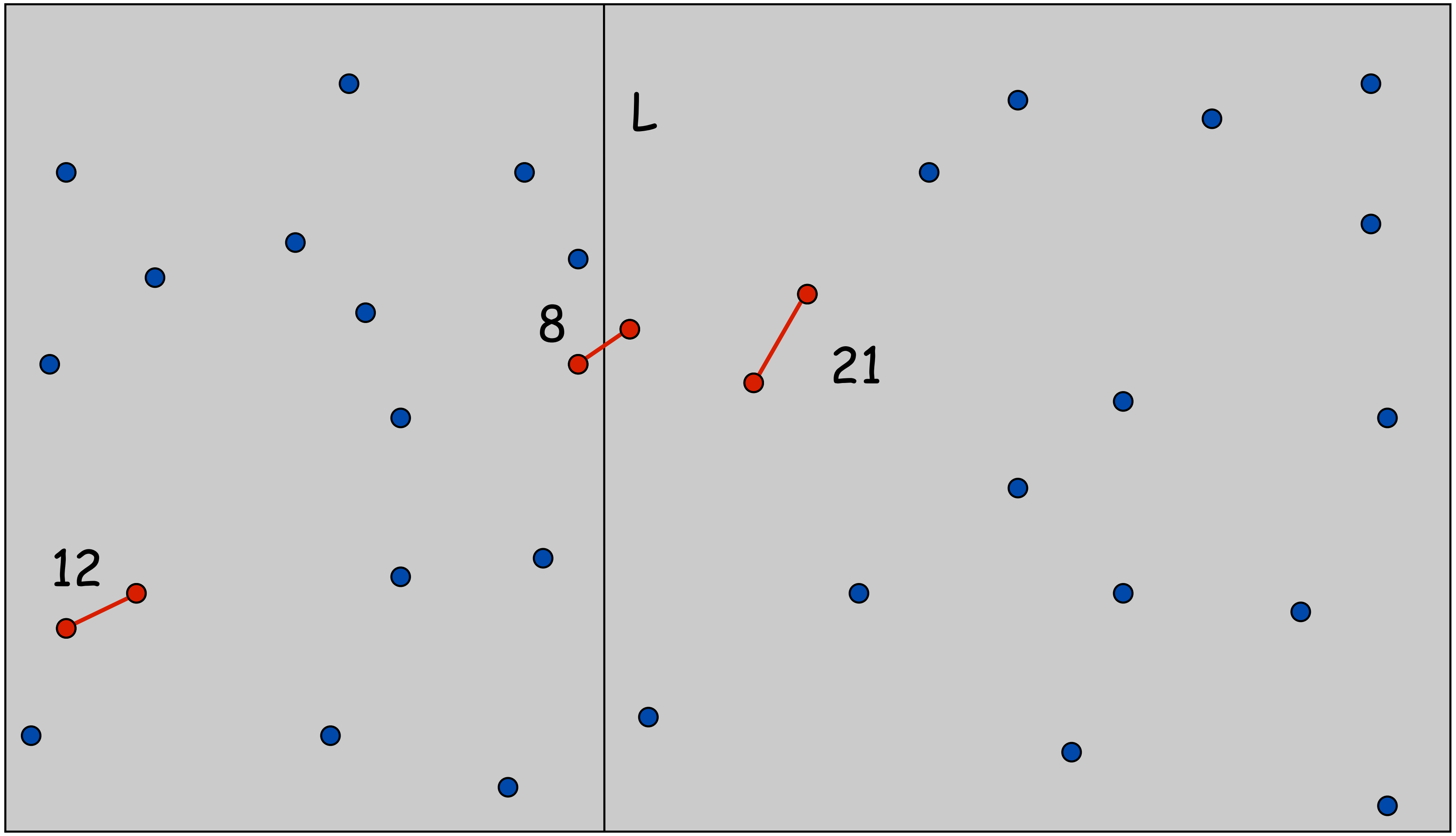
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.



# Closest Pair of Points

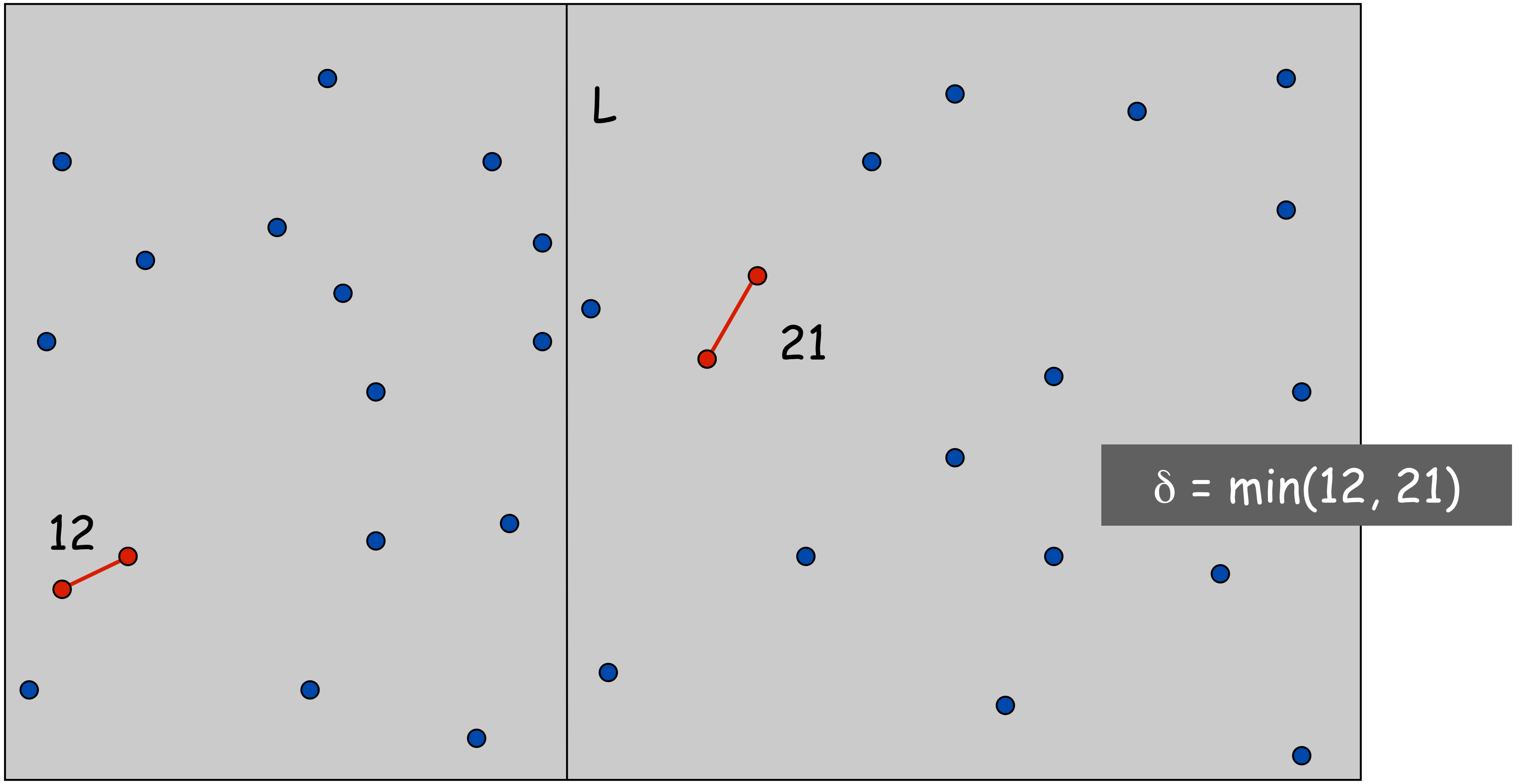
Algorithm.
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side.
- Return best of 3 solutions.

← seems like $\Theta(n^2)$



# Closest Pair of Points

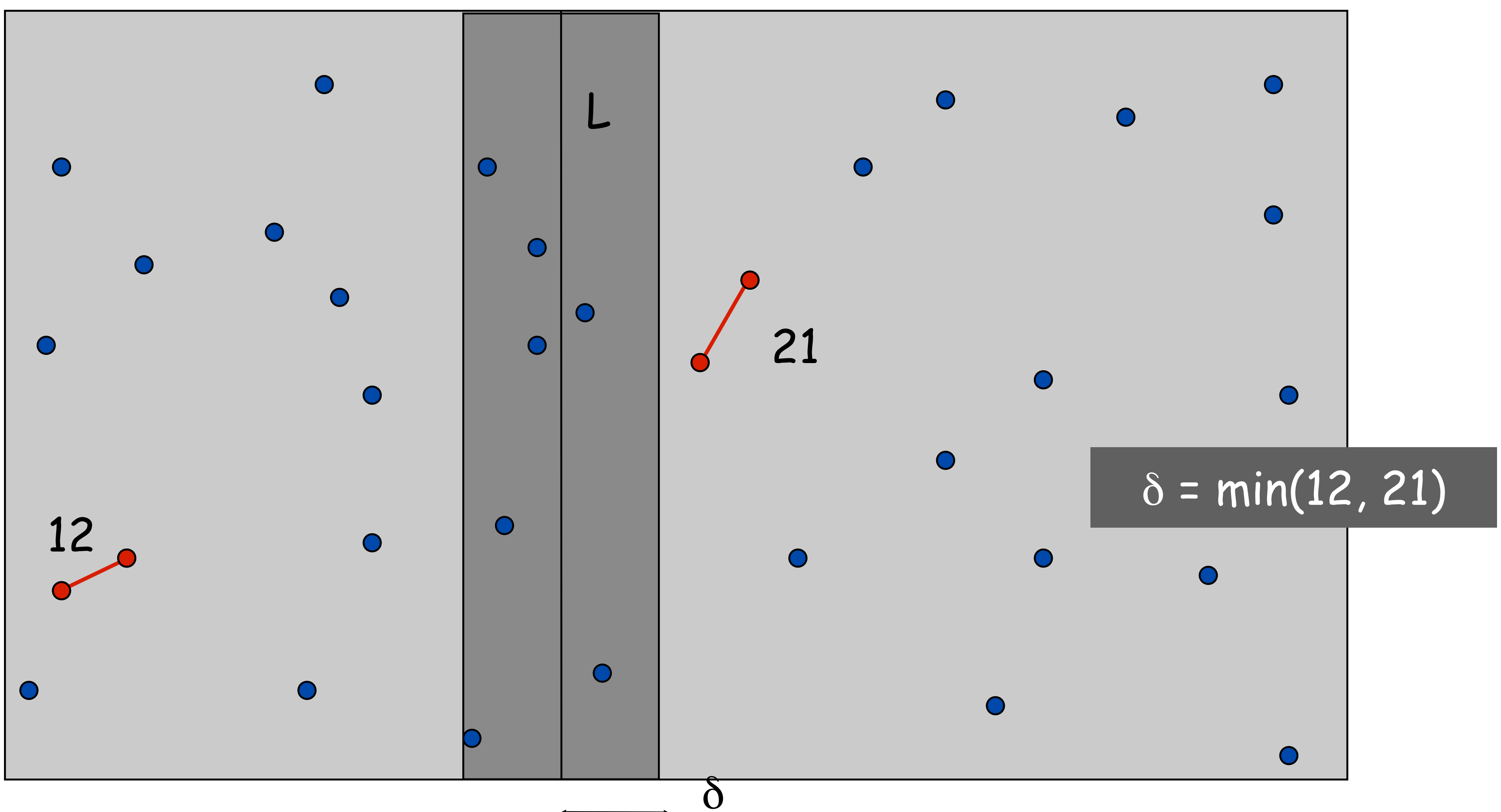
Find closest pair with one point in each side, assuming that distance < $\delta$.



# Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < $\delta$.

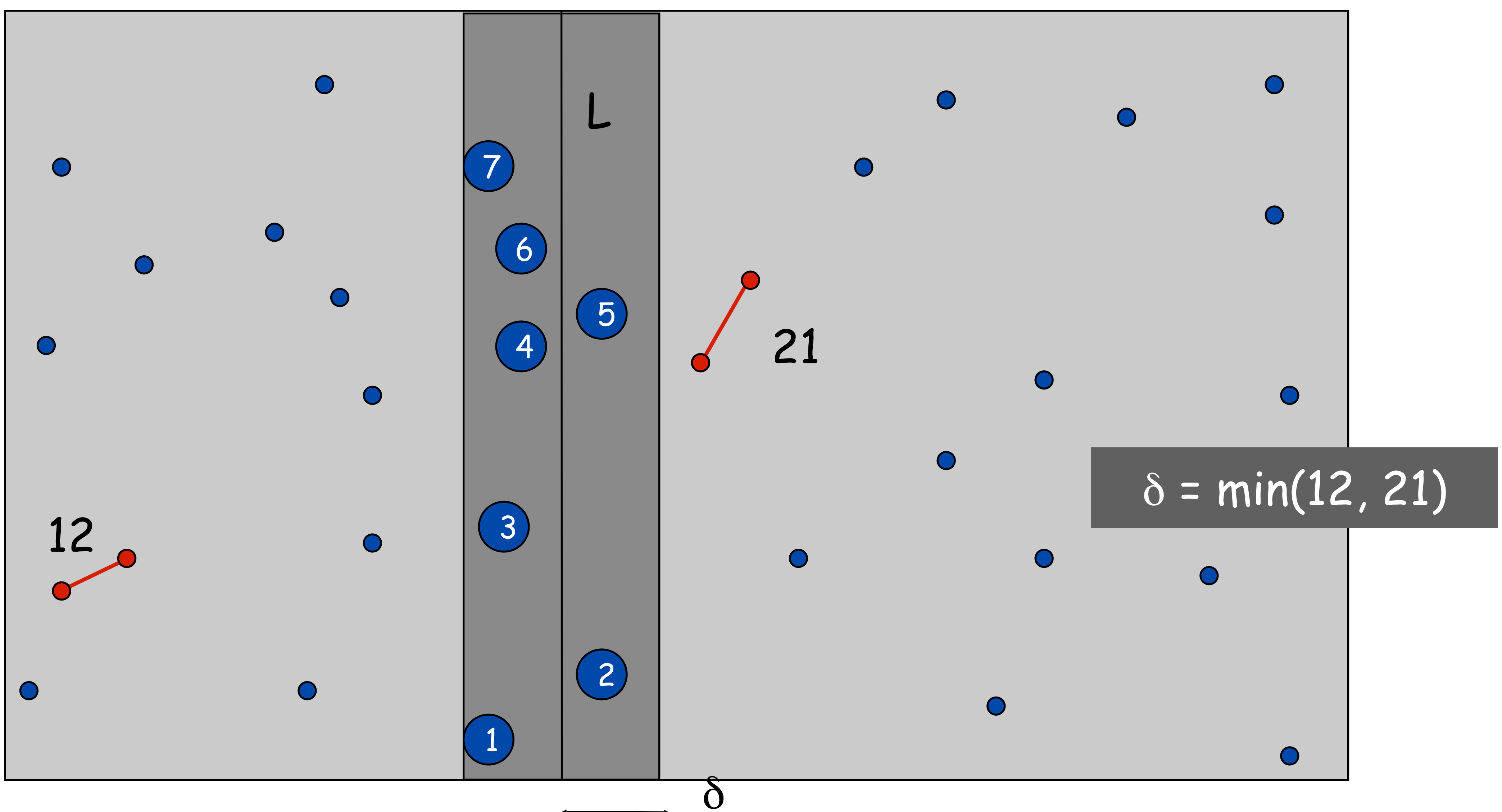
- Observation: only need to consider points within $\delta$ of line L.



# Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < $\delta$.

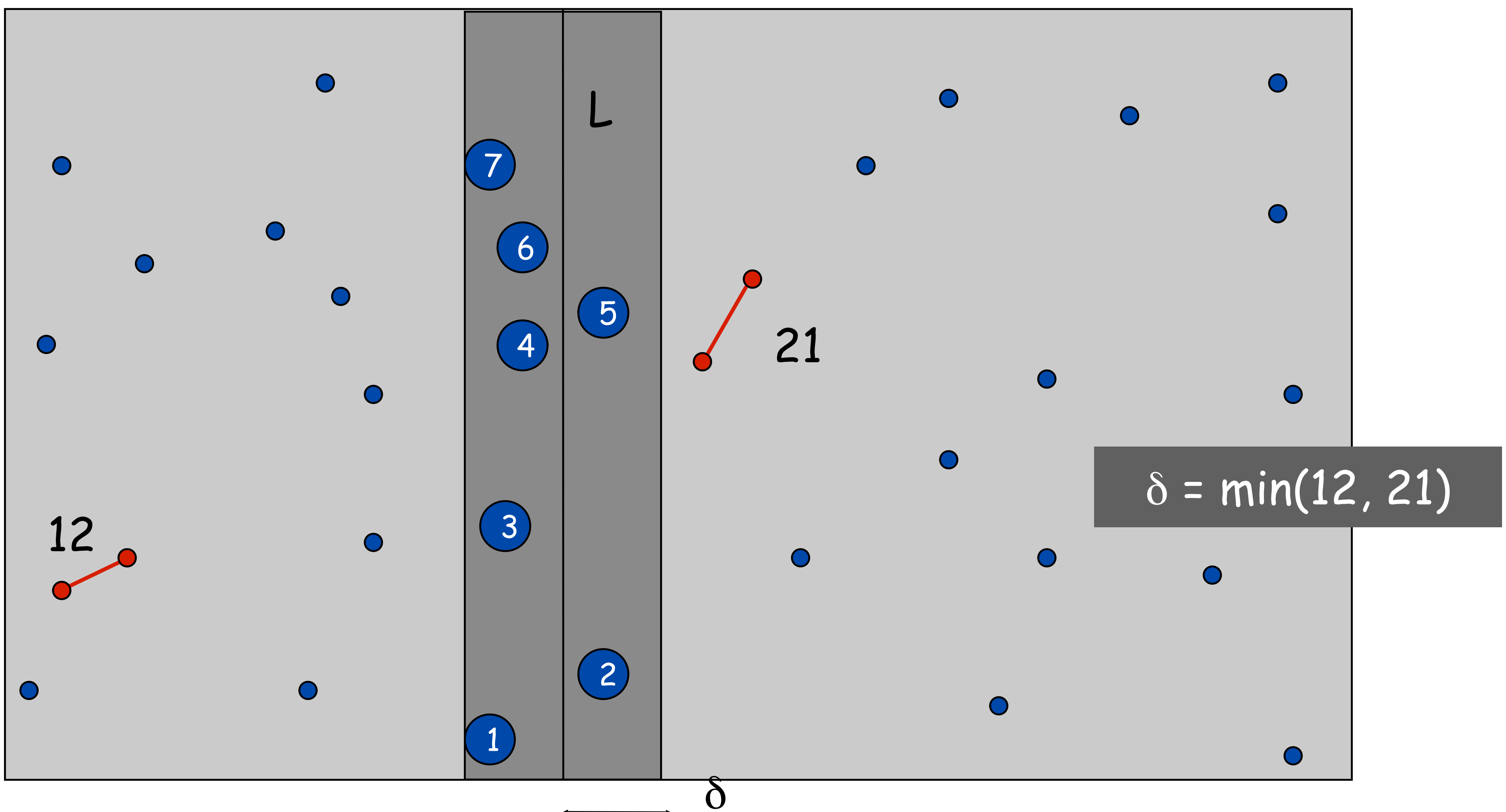
- Observation: only need to consider points within $\delta$ of line L.
- Sort points in $2\delta$-strip by their y coordinate.



# Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < $\delta$.

- Observation: only need to consider points within $\delta$ of line L.
- Sort points in $2\delta$-strip by their y coordinate.
- Only check distances of those within 11 positions in sorted list!



# Closest Pair of Points

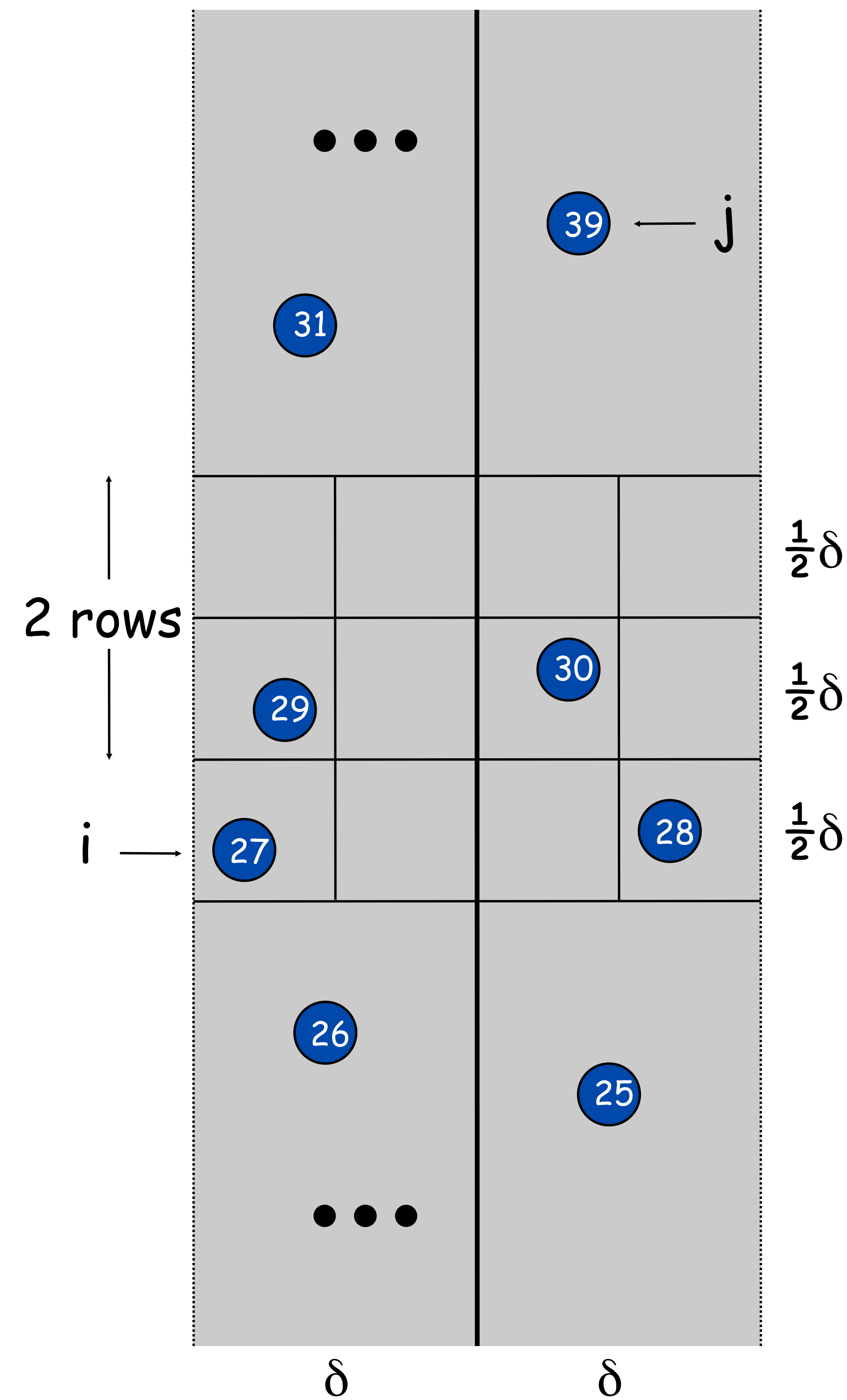
Def. Let $s_i$ be the point in the $2\delta$-strip, with the $i^{th}$ smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between $s_i$ and $s_j$ is at least $\delta$.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪

Fact. Still true if we replace 12 with 7.



# Closest Pair Algorithm

```
Closest-Pair(p₁, …, pₙ) {
   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ₁ = Closest-Pair(left half)
   δ₂ = Closest-Pair(right half)
   δ  = min(δ₁, δ₂)

   Delete all points further than δ from separation line L

   Sort remaining points by y-coordinate.

   Scan points in y-order and compare distance between
   each point and next 11 neighbors. If any of these
   distances is less than δ, update δ.

   return δ.
}
```

$O(n \log n)$

$2T(n / 2)$

$O(n)$

$O(n \log n)$

$O(n)$

# Closest Pair of Points: Analysis

Running time.

$$T(n) \le 2T(n/2) + O(n \log n) \Rightarrow T(n) = O(n \log^2 n)$$

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points in strip from scratch each time.
- Each recursive returns two lists: all points sorted by y coordinate, and all points sorted by x coordinate.
- Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$$