Transmit Power Optimization for Gaussian Vector Broadcast Channels

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***Abstract*— This paper proposes a transmit power optimization algorithm for a Gaussian vector broadcast channel. The optimal power allocation minimizes the sum trace of input covariance matrices of all the users in the Gaussian vector broadcast channel. By duality the transmit power optimization can be transformed into an equivalent convex optimization in a multiple access channel. Recursive equations are derived as sufficient conditions for the minimum transmit power and an iterative algorithm is proposed to solve the equations. Numerical examples show that the iterative algorithm converges very fast and efficiently determines the input covariance matrices for the optimum power allocation.**

I. Introduction

A broadcast channel (BC), where a common base station sends independent information to multiple uncoordinated users, is a typical downlink channel model encountered in multi-user communication. The broadcast channel was first introduced by Cover [1], and the capacity of a BC has been an important research topic since then. For the degraded BC, superposition coding is known to be optimal in the sense that it allows maximum data throughput under a power constraint. Some previous works have focused on the sum capacity of the non-degraded BC as a good capacity bound of the achievable region [2], [3]. The dirty-paper capacity region for a BC is known to be exactly equal to that of the dual multiple access channel (MAC). This duality is first introduced by Yu and Cioffi for the worst-case noise in a BC [4], and the mathematical BC-to-MAC (or MAC-to-BC) transformation is formulated by Vishwanath et al.[3]. By the duality, maximizing the sum rate of a BC is then equivalent to maximizing the sum rate of the dual MAC whose sum power is equal to the BC power constraint. The maximum sum rate problem in a MAC was extensively studied in [5]. The iterative water-filling (IW) proposed in [5] efficiently maximizes the sum rate under individual power constraints. The IW can be modified to solve the sum rate maximization problem under sum power constraint [6], which provides the maximum sum rate of a non-degraded BC by duality .

While most of the research on a BC has been concentrated on sum rate maximization, only some research has been devoted to the inverse problem: how to minimize the transmit power when a data rate tuple for the users in a BC is specified. This power minimization is related to the quality of service (QoS) in a downlink data transmission. The sum rate maximization by [5] cannot always guarantee rate-fairness among users. For the multi-user OFDM case, an efficient algorithm is shown in [7] that dynamically allocates transmit power to guarantee the specific data rate for each user in a MAC. The optimum transmit power in [7] minimizes the sum of the transmit power of all users. This paper extends the result to a Gaussian vector BC using duality. The duality permits conversion of the minimum transmit power problem in a BC to a dual power minimization in a MAC, which is more tractable than the original problem. This paper first derives a convex optimization problem equivalent to the transmit power minimization, and then proposes an algorithm that iteratively solves the optimization. The iterative algorithm converges very fast and solves the transmit power minimization with low computational complexity. The algorithm automatically determines the input covariance matrix for each user in the multiple transmit dimensions.

This paper is organized as follows. Section II shows a Gaussian vector BC and formulates a transmit power optimization problem. Section III derives a set of recursive equations by which the optimum solution can be obtained. An iterative algorithm is implemented and its convergence is shown in section IV. Numerical examples for the proposed algorithms are given in section V, and conclusions are drawn in section VI.

II. Problem Formulation

This paper considers K users in the vector Gaussian BC where the transmitter and the receivers have N antennas. The downlink channel model for user i is then represented as

$$\mathbf{y}_i = H_i^T \mathbf{x} + \mathbf{n}_i, \quad i = 1, \cdots, K \tag{1}$$

where $\mathbf{y}_i$ is the vector output signal, $\mathbf{x}$ is the vector input signal, $\mathbf{n}_i$ is the additive Gaussian noise vector, and H_i^T is the time-invariant channel matrix. The transmitter is assumed to know the channel information perfectly. When each of the users in the BC requires a specific data rate, the transmitter should determine the input covariance matrix for the user to achieve the data rate by decoding the input vector. The optimal strategy for the transmitter is to minimize the transmit power while guaranteeing the required data rate. Then, the optimal covariance matrix can be obtained by solving the following

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optimization problem:

$$\begin{aligned} \text{minimize} \quad & \text{tr}(\sum_{i=1}^{K} S_i^B) \\ \text{subject to} \quad & \frac{1}{2}\log_2 \frac{|I+\sum_{j=1}^{i} H_i^T S_j^B H_i|}{|I+\sum_{j=1}^{i-1} H_i^T S_j^B H_i|} = R_i^B, \\ & i = 1, \cdots, K \end{aligned} \quad (2)$$

where R_i^B is the data rate and S_i^B is the input covariance matrix for user i. S_i^B is an $N \times N$ positive semi-definite matrix and the noise covariance matrix is assumed to be an $N \times N$ identity matrix. The formulation in (2) assumes that the dirty paper coding is employed at the transmitter such that user K is encoded first, user $K-1$ second, and so on. Thus, user 1 who is encoded last sees no interference from the other users. The constraints in (2) are not easily converted to general convex forms, and the optimization problem requires extensive numerical search to find the minimum. However, the duality presented in [3] provides an efficient method of converting the optimization problem into a simple convex form. The channel information of the dual MAC is given by H_i. Using the BC-to-MAC transformation in [3], the rate constraint for user i is then transformed as

$$\frac{1}{2}\log_2 \frac{|I+\sum_{j=i}^{K} H_j S_j^M H_j^T|}{|I+\sum_{j=i+1}^{K} H_j S_j^M H_j^T|} = R_i^M \quad (3)$$

where R_i^M is the dual rate for R_i^B, and S_i^M is the dual covariance matrix for S_i^B. From [3], we know that $R_i^M = R_i^B$ and the sum trace of the input covariance is also preserved, or $\text{tr}(\sum_{i=1}^{K} S_i^M) = \text{tr}(\sum_{i=1}^{K} S_i^B)$.

The dual constraint in (3) is not still tractable. A simple convex form occurs by introducing new variables $\tilde{R}_i^M = \sum_{j=i}^{K} R_j^M$. The partial rate sum $\tilde{R}_i^M$ is a concave function of input covariance matrices, which is a unique property of a MAC. Finally, the equivalent convex optimization problem for the transmit power minimization is:

$$\begin{aligned} \text{minimize} \quad & \text{tr}(\sum_{i=1}^{K} S_i^M) \\ \text{subject to} \quad & \frac{1}{2}\log_2 |I+\sum_{j=i}^{K} H_j S_j^M H_j^T| = \tilde{R}_i^M, \\ & i = 1, \cdots, K. \end{aligned} \quad (4)$$

This dual form can be interpreted as the transmit sum power minimization in a MAC where multiple transmitters send independent information to a common receiver, and can be considered as an uplink transmit power optimization. Since the objective function is linear and the constraints are concave, the optimization problem in (4) belongs to a class of convex programming for which an efficient numerical method exists for the solution [8]. However, section IV proposes a faster iterative algorithm to solve the optimization by investigating the margin adaptive (MA) water-filling technique. From now on, the superscripts in R_i^M and S_i^M are removed if there is no ambiguity.

III. Optimum Transmit Power

This section shows a sufficient condition for the optimal input covariance matrices and derives an efficient algorithm to solve the optimization problem. As proposed in [5], the rate adaptive (RA) water-filling can be extended to the rate maximization in a MAC. The iterative water-filling for a MAC greatly reduces the computational complexity by exploiting the problem structure. Similarly, another iterative algorithm is developed to solve the transmit power minimization based on the MA water-filling for a single-user transmission.

Let $L(\{S_i\},\{\lambda_i\})$ be the Lagrangian for the optimization problem in (4), which is given by

$$\begin{aligned} L(\{S_i\},\{\lambda_i\}) &= \text{tr}(\sum_{i=1}^{K} S_i) \\ & - \sum_{i=1}^{K} \lambda_i (\frac{1}{2}\log_2 |I+\sum_{j=i}^{K} H_j S_j H_j^T| - \tilde{R}_i) \end{aligned} \quad (5)$$

where λ_i's are the Lagrange multipliers. Deriving partial derivatives with respect to S_i, and applying Karush-Khun-Tucker (KKT) conditions, $\frac{\partial L}{\partial S_i} = 0$, we can obtain

$$I - \frac{1}{2}\sum_{j=1}^{i} \lambda_j H_i^T A_j^{-1} H_i = 0, \quad (6)$$

where $A_j = I + \sum_{l=j}^{K} H_l S_l H_l^T$. Since the optimization problem is convex, S_i's obtained by solving (6) are the input covariance matrices that give the minimum transmit power. However, explicit solutions to S_i's are not possible in (6), thus the KKT conditions should be modified to derive tractable equations for the input covariance matrices. $H_i H_i^T$ is assumed non-singular, which is generally true in a MIMO wireless BC, since the probability that a randomly-chosen $H_i H_i^T$ is singular is almost zero. With the assumption, the KKT conditions can be rewritten as

$$A_i = H_i S_i H_i^T + (I + \sum_{j=i+1}^{K} H_j S_j H_j^T) = (\frac{1}{2}\sum_{j=1}^{i} \lambda_j) P_i \quad (7)$$

where

$$P_i = \{(H_i H_i^T)^{-1} + A_i^{-1} \sum_{j=1}^{i-1} (H_j S_j H_j^T)(H_j H_j^T)^{-1}\}^{-1}. \quad (8)$$

The detailed derivations of (7) and (8) are shown in the Appendix. Note that A_i is the sum of the signal and the interference for user i, and P_i is the modified channel gain matrix which consists of not only the channel gain matrix $H_i H_i^T$ of user i but also those of the other users who are decoded earlier than user i.

Using all the above equations, the optimization problem can be equivalently re-formulated as

$$\begin{aligned} \text{find} \quad & S_i \\ \text{subject to} \quad & H_i S_i H_i^T + N_i = \tilde{\lambda}_i P_i \\ & \frac{1}{2}\log_2 \frac{|H_i S_i H_i^T + N_i|}{|N_i|} = R_i, \end{aligned} \quad (9)$$

where $N_i = I + \sum_{l=i+1}^{K} H_l S_l H_l^T$ and $\tilde{\lambda}_i = \frac{1}{2}\sum_{j=1}^{i} \lambda_j$, respectively. The solution to (9) provides the optimal input covariance for user i that minimizes the transmit power. Note that, if $P_i = H_i H_i^T$, then (9) is a simple extension of MA water-filling to a multi-user case, and can be solved by iteratively applying MA water-filling. However, P_i also contains A_i, thus, (9) is a recursive equation on A_i. This indicates an interesting fact that by (9) the later-encoded users transmit more power with the modified channel gain matrices (P_i's) than with the original channel gain matrices ($H_i H_i^T$'s), while the earlier-encoded users transmitting less power. However, the sum of the individual power is always minimized by (9).

IV. Iterative Transmit Power Minimization

Even though the single-user MA water-filling cannot directly apply, it can be partially exploited in solving the recursive equation. Since there is no closed-form solution to (9) in general, an iterative method should be employed. For fixed input covariance matrices $\{S_1, \cdots, S_K\}$ at a certain stage of iteration, the next input covariance matrix S_i is determined as follows. By eigenvalue decomposition $N_i = Q_i \Delta_i Q_i^T$, where Q_i is an orthogonal matrix $Q_i Q_i^T = I$, and Δ_i is a diagonal matrix of eigenvalues. Define $\hat{H}_i = \Delta_i^{-\frac{1}{2}} Q_i^T H_i$. Then the constraints in (9) can be rewritten as

$$\hat{H}_i S_i \hat{H}_i^T + I = \tilde{\lambda}_i (Q_i \Delta_i^{\frac{1}{2}})^T P_i (Q_i \Delta_i^{\frac{1}{2}}) \qquad (10)$$

$$\frac{1}{2}\log_2 |\hat{H}_i S_i \hat{H}_i^T + I| = R_i. \qquad (11)$$

Our objective is then to find S_i and $\tilde{\lambda}_i$ that simultaneously satisfy (10) and (11). The single-user MA water-filling can determine λ_i under the data rate constraint R_i. Define $\hat{P}_i = (Q_i \Delta_i^{\frac{1}{2}})^T P_i (Q_i \Delta_i^{\frac{1}{2}})$. Then $\hat{P}_i$ is also positive definite, and can be decomposed as $\hat{P}_i = F_i \Sigma_i F_i^T$, where F_i is an orthogonal matrix $F_i F_i^T = I$, and Σ_i is a diagonal matrix of eigenvalues. Note that $\Sigma_i = \mathrm{diag}\{\sigma_{i1}, \cdots, \sigma_{iN}\}$ is diagonal channel gain matrix when $\hat{P}_i$ is regarded modified channel gain matrix. Applying MA water-filling on $\{\sigma_{i1}, \cdots, \sigma_{iN}\}$, we can obtain the water-filling level $\tilde{\lambda}_i$ and the rate vector $\{r_{i1}, \cdots, r_{iN}\}$ such that

$$\begin{aligned} \frac{2^{2r_{ij}}}{\sigma_{ij}} &= \tilde{\lambda}_i, \quad \text{if } \frac{1}{\sigma_{ij}} < \tilde{\lambda}_i \\ r_{ij} &= 0, \quad \text{if } \frac{1}{\sigma_{ij}} \geq \tilde{\lambda}_i \end{aligned} \qquad (12)$$

where $\sum_{j=1}^{N} r_{ij} = R_i$. Define $\hat{S}_i = \mathrm{diag}\{2^{2r_{i1}}, \cdots, 2^{2r_{iN}}\}$. Then, $\hat{S}_i = \tilde{\lambda}_i \Sigma_i$ for the non-zero r_{ij}, and the optimum covariance is given by

$$S_i = \hat{H}_i^{-1} (F_i \hat{S}_i F_i^T - I) \hat{H}_i^{-T} \qquad (13)$$

where $\hat{H}_i^{-T}$ is the inverse of the transposed matrix $\hat{H}_i^T$, and H_i is assumed non-singular. By this process, S_i that satisfies the KKT condition for user i can be founded. Once S_i is updated, the process is applied to the next user such that S_{i-1} also satisfies the KKT condition. Continue the process iteratively from user K to user 1 until the sum trace converges. Then the set of $\{S_i\}$ must satisfy the KKT conditions. By the sufficiency of the KKT conditions, $\{S_i\}$ must be the optimal covariance matrices for the minimum transmit power. More specifically, the iterative algorithm can be described as follows:

Algorithm 1: Iterative Transmit Power Minimization.

initialize $S_i =$ null matrix, $i = 1, \cdots, K$
repeat from $i = K$ to 1
compute P_i
solve the equation in (9) to obtain an updated S_i
until there is no further decrease in the sum trace of $\{S_i\}$

The iterative algorithm converges fast. The speed of convergence usually depends on the number of users, channel matrices, and the number of antennas. The convergence can be proved as a special case of the block coordinate descent method in [9]. It should be noticed that the proposed algorithm can only be applied to square channel matrices. For a general channel matrix such as an $N \times M$ $(N > M)$ matrix, where N and M is the number of receive and transmit antennas, respectively, there is no optimal solution so far. However, the proposed algorithm can still be employed. By introducing $(N - M)$ virtual transmit antennas at the transmitter and assigning small channel gains from the virtual antennas to the receive antennas, we can construct a square $N \times N$ channel matrix. The virtual channel gains should be assigned in such a way that the condition number of the square matrix is not too big. Also, they should be small enough so that the MA water-filling in the iterative algorithm does not assign any transmit power to the virtual transmit antennas. For the modified square matrices in a BC, the iterative algorithm provides input covariance matrices. If the columns and rows that belong to the virtual transmit antennas are discarded, the sub-optimal input covariance matrix for the $N \times M$ channel matrix are obtained.

Note also that the minimum transmit power depends on the encoding order as in the multi-user OFDM cases [7]. The ordering dependency in transmit power minimization becomes more noticeable especially when the channel gain matrices among users show significant differences in norm. Thus, the transmit power minimization consists of two steps; the first step is to find the best encoding order among users and the second step is to solve the optimization problem by using the proposed iterative algorithm subject to the predetermined encoding order. Finding the best ordering may require exhaustive search, but a simple method for determining a good encoding order can be suggested: encode first the user who has the lowest channel gain and last the user who has the highest channel gain. For a channel matrix, Frobenius norm can be employed as a channel gain in determining the encoding order. By this method optimum input covariance matrices can be obtained in most cases without applying the iterative algorithm for all the possible $K!$ orderings.

V. NUMERICAL EXAMPLES

This section provides numerical examples to simulate the proposed iterative algorithm in a Gaussian vector BC. First, a 3-user BC where the transmitter and the receivers have two antennas is considered. The channel matrices and the desired data rates are given as below:

- Channel Matrices :

$$H_1 = \begin{bmatrix} 5 & 5 \\ 5 & 3 \end{bmatrix} \quad H_2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

- Desired Data Rates : $R_1 = R_2 = R_3 = 1$ (bit/dim)
- Noise Covariance Matrix : $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Suppose the encoding order is descending (user 3 is encoded first, user 2 is encoded second, etc.). By applying the proposed iterative algorithm, the optimum covariance matrices for the dual MAC are given by

$$S_1^M = \begin{bmatrix} 1.022 & 1.042 \\ 1.042 & 1.064 \end{bmatrix} \quad S_2^M = \begin{bmatrix} 0.676 & -0.202 \\ -0.202 & 8.435 \end{bmatrix}$$
$$S_3^M = \begin{bmatrix} 0.618 & 0.585 \\ 0.585 & 0.554 \end{bmatrix}.$$

Then, by the MAC-to-BC transformation in [3], we can find the corresponding BC covariance matrices

$$S_1^B = \begin{bmatrix} 0.324 & -0.02 \\ -0.02 & 0 \end{bmatrix} \quad S_2^B = \begin{bmatrix} 4.479 & -3.829 \\ -3.829 & 3.376 \end{bmatrix}$$
$$S_3^B = \begin{bmatrix} 1.5 & 2.008 \\ 2.008 & 2.690 \end{bmatrix}.$$

We can check that $\{S_i^B\}$ satisfy the rate constraints:

$$R_i^B = \frac{1}{2}\log_2 \frac{|I + \sum_{j=1}^{i} H_i^T S_j^B H_i|}{|I + \sum_{j=1}^{i-1} H_i^T S_j^B H_i|} = 2, \quad \forall i \tag{14}$$

where R_i^B is twice of the desired data rate (bit/dim), since the receivers have two transmit dimensions (two transmit antennas). Fig. 1 shows the convergence of sum trace of covariance matrices as the iteration time increases. The minimum transmit power is 12.37 in this case.

When the encoding order for the same channel matrices is changed, the transmit power minimization converges to a different value. Let the encoding order be the sequence of user 1, user 3, and then user 2 (ordering 1-3-2, for short). Then, the optimum covariance matrices are different from the ones obtained in the previous case and the minimum sum trace is also changed. The optimum covariance matrices with the new encoding order are given by

$$S_1^B = \begin{bmatrix} 5.752 & 5.659 \\ 5.659 & 5.568 \end{bmatrix} \quad S_2^B = \begin{bmatrix} 6.484 & -4.864 \\ -4.864 & 3.648 \end{bmatrix}$$
$$S_3^B = \begin{bmatrix} 0.790 & -1.708 \\ -1.708 & 3.692 \end{bmatrix}.$$

Fig. 1 also shows the convergence behavior for other orderings. Table I compares the minimum transmit power depending on the order. It is clear that the best ordering that provides the minimum transmit power is the sequence of user 2, user 3, and then user 1. This is exactly the ascending order of Frobenius norm of the channel gain matrices.

TABLE I
ORDERING AND THE MINIMUM TRANSMIT POWER

Ordering	1-2-3	1-3-2	2-3-1
Minimum Transmit Power	24.39	25.93	9.95
Ordering	2-1-3	3-1-2	3-2-1
Minimum Transmit Power	20.21	13.55	12.37

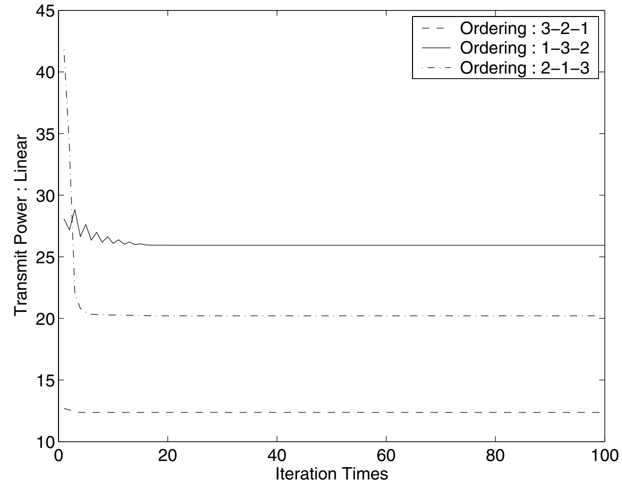

Fig. 1. Convergence of sum trace for some decoding orders: 3 users, 2×2 MIMO BC case.

For the second numerical example, 3×2 non-square channel matrices for a 2-user BC are considered. In this case, the receivers have more antennas than the transmitter. The channel matrices are given by

$$H_1 = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 2 \end{bmatrix} \quad H_2 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

The rate constraints and the noise matrix are the same as in the previous example. Now, with a virtual antenna introduced at the transmitter and small channel gains $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.001$ for the antenna, square channel matrices can be constructed as

$$\tilde{H}_1 = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 2 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \end{bmatrix} \quad \tilde{H}_2 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \end{bmatrix}.$$

Apply the iterative algorithm to a BC consisting of $\tilde{H}_1$ and $\tilde{H}_2$ with user 2 being encoded first (ordering 2-1, for short). The optimal input matrices are then given by

$$\tilde{S}_1^B = \begin{bmatrix} 0.02 & -0.124 & 0 \\ -0.124 & 0.779 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\tilde{S}_2^B = \begin{bmatrix} 1.651 & 0.267 & 0.001 \\ 0.267 & 0.94 & 0.001 \\ 0.001 & 0.001 & 0 \end{bmatrix}.$$

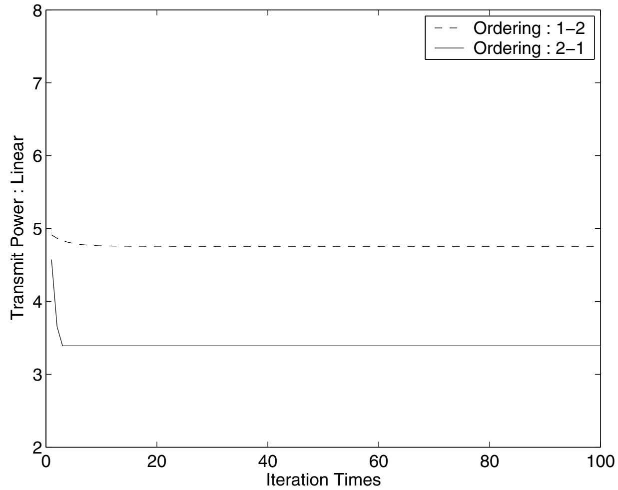

Fig. 2. Convergence of sum trace for two decoding orders: 2 users, 3×2 MIMO BC case.

Once the third rows and columns of $\tilde{S}_1^B$ and $\tilde{S}_2^B$ are removed, the sub-optimal input matrices for the 3×2 non-square channel case are finally obtained as below,

$$S_1^B = \begin{bmatrix} 0.02 & -0.124 \\ -0.124 & 0.779 \end{bmatrix} \quad S_2^B = \begin{bmatrix} 1.651 & 0.267 \\ 0.267 & 0.94 \end{bmatrix}.$$

Using (14), the date rates for S_1^B and S_2^B are given by $R_1^B = R_2^B = 2$, which is twice of the desired data rate (bit/dim), since the receivers have two transmit dimensions. This shows that S_1^B and S_2^B are very close to optimum and provide the desired data rates, though they are sub-optimal. Fig. 2 shows the convergence behavior for the 3×2 BC case.

VI. Conclusions

In conclusion, the iterative algorithm proposed in this paper solves transmit power minimization problem in a Gaussian vector BC under the rate constraints for the users. The iterative algorithm not only economizes the transmit power resources but also reduces the interference to the adjacent cells. The BC-MAC duality also indicates that the proposed algorithm can be directly applied to the minimum transmission power problem in a Gaussian vector MAC. Thus, the uplink and the downlink transmit power control can be performed with the iterative algorithm.

The iterative algorithm also provides sub-optimal solutions to non-square channel matrices. The transmit power optimization for general channel matrices and the best encoding order for transmit power minimization are not still clear.

Appendix

The derivation of (7) is given as follows. Using the assumption that $H_i H_i^T$ is non-singular, the KKT conditions in (6) can be rewritten as

$$\begin{aligned} \frac{1}{2}\lambda_1 I &= A_1 (H_1 H_1^T)^{-1} \\ \frac{1}{2}\lambda_i I &= A_i \{(H_i H_i^T)^{-1} - (H_{i-1} H_{i-1}^T)^{-1}\}, \ i \geq 2. \end{aligned} \quad (15)$$

For the simplicity of computation, define an inverse channel gain matrix G_i^{-1}

$$\begin{aligned} G_1^{-1} &= (H_1 H_1^T)^{-1} \\ G_i^{-1} &= (H_i H_i^T)^{-1} - (H_{i-1} H_{i-1}^T)^{-1}, \quad i \geq 2. \end{aligned} \quad (16)$$

To apply the MA water-filling technique for the transmit power optimization problem, a positive definite gain matrix is desired. However, in (16), G_i^{-1}'s are not always positive definite. Thus, the KKT conditions are modified again to obtain a positive definite gain matrix. From (15) and (16),

$$\begin{aligned} &(\frac{1}{2}\sum_{j=1}^{i} \lambda_j) I = \sum_{j=1}^{i} A_j G_j^{-1} \\ = \ & A_i (G_i^{-1} + A_i^{-1} \sum_{j=1}^{i-1} A_j G_j^{-1}) \\ = \ & A_i \{(H_i H_i^T)^{-1} - \sum_{j=1}^{i-1} G_j^{-1} + A_i^{-1} \sum_{j=1}^{i-1} A_j G_j^{-1}\} \\ = \ & A_i \{(H_i H_i^T)^{-1} + \sum_{j=1}^{i-1} (A_i^{-1} A_j - I) G_j^{-1}\} \\ = \ & A_i [(H_i H_i^T)^{-1} + A_i^{-1}(A_1 - A_i)(H_1 H_1^T)^{-1} \\ & + A_i^{-1} \sum_{j=2}^{i-1} \{A_j - A_i\}\{(H_j H_j^T)^{-1} - (H_{j-1} H_{j-1}^T)^{-1}\}] \\ = \ & A_i \{(H_i H_i^T)^{-1} + A_i^{-1} \sum_{j=1}^{i-1} (A_j - A_{j+1})(H_j H_j^T)^{-1}\} \\ = \ & A_i \{(H_i H_i^T)^{-1} + A_i^{-1} \sum_{j=1}^{i-1} (H_j S_j H_j^T)(H_j H_j^T)^{-1}\}. \end{aligned} \quad (17)$$

Then, the modified positive definite gain matrix P_i is defined by (8). From (17), we obtain $A_i P_i^{-1} = (\frac{1}{2}\sum_{j=1}^{i} \lambda_j) I$, which is the recursive equation in (7).

References

[1] T. Cover, "Broadcast channels," *IEEE Trans. on Information Theory,* vol. 18, pp. 2-14, Jan. 1972.

[2] G. Caire and S. Shamai, "On the achievable throughput of a multiantenna Gaussian broadcast channel," *IEEE Trans. on Information Theory,* vol. 49, pp. 1691-1706, Jul. 2003.

[3] S. Vishwanath, N. Jindal, and A. Goldsmith, "Duality, achievable rates, and sum-rate capacity of Gaussian MIMO broadcast channels," *IEEE Trans. on Information Theory,* vol. 49, pp. 2658-2668, Oct. 2003.

[4] W. Yu and J. Cioffi, "Sum capacity of Gaussian vector broadcast channels," *IEEE Trans. on Information Theory,* vol. 50, pp. 1875 - 1892, Sep. 2004.

[5] W. Yu, W. Rhee, S. Boyd, and J. Cioffi, "Iterative water-filling for Gaussian vector multiple access channels," *IEEE Trans. on Information Theory,* vol. 50, pp. 145-152, Jan. 2004.

[6] N. Jindal, W. Rhee, S. Jafar, S. Vishwanath, and A. Goldsmith, "Sum power iterative water-filling for multi-antenna Gaussian broadcast channels," *submitted to IEEE Trans. on Information Theory* .

[7] J. Oh, S. Kim, and J. Cioffi, "Optimum power allocation and control for OFDM in multiple access channles," *Proc. 60th IEEE Veh. Tech. Conf.*, Sep. 2004.

[8] S. Boyd and L. Vandenberghe, *Convex Optimization.* Cambridge University Press, 2004.

[9] D. Berteskas, *Nonlinear Programming.* Athena Scientific, 1999.