

On the 2-Colorability of Random Hypergraphs

Dimitris Achlioptas and Cristopher Moore*

¹ Microsoft Research, Redmond, Washington optas@microsoft.com

² Computer Science Department, University of New Mexico, Albuquerque and the Santa Fe Institute, Santa Fe, New Mexico moore@cs.unm.edu

Abstract. A 2-coloring of a hypergraph is a mapping from its vertices to a set of two colors such that no edge is monochromatic. Let $H_k(n, m)$ be a random k -uniform hypergraph on n vertices formed by picking m edges uniformly, independently and with replacement. It is easy to show that if $r \geq r_c = 2^{k-1} \ln 2 - (\ln 2)/2$, then with high probability $H_k(n, m = rn)$ is not 2-colorable. We complement this observation by proving that if $r \leq r_c - 1$ then with high probability $H_k(n, m = rn)$ is 2-colorable.

1 Introduction

For an integer $k \geq 2$, a k -uniform hypergraph H is an ordered pair $H = (V, E)$, where V is a finite non-empty set, called the set of *vertices* of H , and E is a family of distinct k -subsets of V , called the *edges* of H . For general hypergraph terminology and background see [5]. A 2-coloring of a hypergraph $H = (V, E)$ is a partition of its vertex set V into two (color) classes so that no edge in E is monochromatic. A hypergraph is 2-colorable if it admits a 2-coloring.

The property of 2-colorability was introduced and studied by Bernstein [6] in the early 1900s for infinite hypergraphs. The 2-colorability of finite hypergraphs, also known as “Property B” (a term coined by Erdős in reference to Bernstein), has been studied for about eighty years (e.g. [4,10,11,15,16,19,20]). For $k = 2$, i.e. for graphs, the problem is well understood since a graph is 2-colorable if and only if it has no odd cycle. For $k \geq 3$, though, much less is known and deciding the 2-colorability of k -uniform hypergraphs is NP-complete [17].

In this paper we discuss the 2-colorability of random k -uniform hypergraphs for $k \geq 3$. (For the evolution of odd cycles in random graphs see [12].) Let $H_k(n, m)$ be a random k -uniform hypergraph on n vertices, where the edge set is formed by selecting uniformly, independently and with replacement m out of all possible $\binom{n}{k}$ edges. We will study asymptotic properties of $H_k(n, m)$ when $k \geq 3$ is arbitrary but fixed while n tends to infinity. We will say that a hypergraph property A holds *with high probability* (w.h.p.) in $H_k(n, m)$ if $\lim_{n \rightarrow \infty} \Pr[H_k(n, m) \text{ has } A] = 1$. The main question in this setting is:

As m is increased, when does $H_k(n, m)$ stop being 2-colorable?

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It is popular to conjecture that the transition from 2-colorability to non-2-colorability is *sharp*. That is, it is believed that for each $k \geq 3$, there exists a constant r_k such that if $r < r_k$ then $H_k(n, m = rn)$ is w.h.p. 2-colorable, but if $r > r_k$ then w.h.p. $H_k(n, m = rn)$ is not 2-colorable. Determining r_k is a challenging open problem, closely related to the satisfiability threshold conjecture for random k -SAT. Although r_k has not been proven to exist, we will take the liberty of writing $r_k \geq r^*$ to denote that for $r < r^*$, $H_k(n, rn)$ is 2-colorable w.h.p. (and analogously for $r_k \leq r^*$).

A relatively recent result of Friedgut [13] supports this conjecture as it gives a *non-uniform* sharp threshold for hypergraph 2-colorability. Namely, for each $k \geq 3$ there exists a *sequence* $r_k(n)$ such that if $r < r_k(n) - \epsilon$ then w.h.p. $H_k(n, rn)$ is 2-colorable, but if $r > r_k(n) + \epsilon$ then w.h.p. $H_k(n, rn)$ is not 2-colorable. We will find useful the following immediate corollary of this sharp threshold.

Corollary 1. *If*

$$\liminf_{n \rightarrow \infty} \Pr[H_k(n, r^*n) \text{ is 2-colorable}] > 0 \text{ ,}$$

then for $r < r^$, $H_k(n, rn)$ is 2-colorable w.h.p.*

Alon and Spencer [3] were the first to give bounds on the potential value of r_k . In particular, they observed that the expected number of 2-colorings of $H_k(n, m = rn)$ is $o(1)$ if $2(1 - 2^{1-k})^r < 1$, implying

$$r_k < 2^{k-1} \ln 2 - \frac{\ln 2}{2} \text{ .} \tag{1}$$

Their main contribution, though, was providing a lower bound on r_k . Specifically, by applying the Lovász Local Lemma, they were able to show that if $r = c2^k/k^2$ then w.h.p. $H_k(n, rn)$ is 2-colorable, for some small constant $c > 0$.

In [1], Achlioptas, Kim, Krivelevich and Tetali reduced the asymptotic gap between the upper and lower bounds of [3] from order k^2 to order k . In particular, they proved that there exists a constant $c > 0$ such that if $r \leq c2^k/k$ then a simple, linear-time algorithm w.h.p. finds a 2-coloring of $H_k(n, rn)$. Their algorithm was motivated by algorithms for random k -SAT due to Chao and Franco [7] and Chvátal and Reed [8]. In fact, those algorithms give a similar $\Omega(2^k/k)$ lower bound on the random k -SAT threshold which, like r_k , can also be easily bounded as $O(2^k)$.

Very recently, the authors eliminated the gap for the random k -SAT threshold, determining its value within a factor of two [2]. The proof amounts to applying the “second moment” method to the set of satisfying truth assignments whose complement is also satisfying. Alternatively, one can think of this as applying the second moment method to the number of truth assignments under which every k -clause contains at least one satisfied literal *and* at least one unsatisfied literal, i.e. which satisfy the formula when interpreted as a random instance of Not-All-Equal k -SAT (NAE k -SAT).

Here we extend the techniques of [2] and apply them to hypergraph 2-colorability. This allows us to determine r_k within a small additive constant.

Theorem 1. *For every $\epsilon > 0$ and all $k \geq k_0(\epsilon)$,*

$$r_k \geq 2^{k-1} \ln 2 - \frac{\ln 2}{2} - \frac{1 + \epsilon}{2} .$$

Our method actually yields an explicit lower bound for r_k for each value of k as the solution to a simple equation (yet one without a pretty closed form, hence Theorem 1). Below we compare this lower bound to the upper bound of (1) for small values of k . The gap converges to $1/2$ rather rapidly.

Table 1. Upper and lower bounds for r_k

| k | 3 | 4 | 5 | 7 | 9 | 11 | 12 |
|-------------|-------|-------|--------|--------|---------|---------|----------|
| Lower bound | 3/2 | 49/12 | 9.973 | 43.432 | 176.570 | 708.925 | 1418.712 |
| Upper bound | 2.409 | 5.191 | 10.740 | 44.014 | 177.099 | 709.436 | 1419.219 |

2 Second Moment and NAE k -SAT

We prove Theorem 1 by applying the following version of the second moment method (see Exercise 3.6 in [18]) to the number of 2-colorings of $H_k(n, m = rn)$.

Lemma 1. *For any non-negative integer-valued random variable X ,*

$$\Pr[X > 0] \geq \frac{\mathbf{E}[X]^2}{\mathbf{E}[X^2]} . \tag{2}$$

In particular, if X is the number of 2-colorings of $H_k(n, m = rn)$, we will prove that for all $\epsilon > 0$ and all $k \geq k_0(\epsilon)$, if $r = 2^{k-1} \ln 2 - \ln 2/2 - (1 + \epsilon)/2$ then there exists some constant $C = C(k)$ such that

$$\mathbf{E}[X^2] < C \times \mathbf{E}[X]^2 .$$

By Lemma 1, this implies $\Pr[X > 0] = \Pr[H_k(n, rn) \text{ is 2-colorable}] > 1/C$. Theorem 1 follows by invoking Corollary 1.

This approach parallels the one taken recently by the authors for random NAE k -SAT [2]. Naturally, what differs is the second-moment calculation which here is *prima facie* significantly more involved.

We start our exposition by outlining the NAE k -SAT calculation of [2]. This serves as a warm up for our calculations and allows us to state a couple of useful

lemmata from [2]. We then proceed to outline the proof of our main result, showing the parallels with NAE k -SAT and reducing the proof of Theorem 1 to the proof of three independent lemmata.

The first such lemma is specific to hypergraph 2-colorability and expresses $\mathbf{E}[X^2]$ as a multinomial sum. The second one is a general lemma about bounding multinomial sums by a function of their largest term and is perhaps of independent interest. It generalizes Lemma 2 of [2], which we state below. After applying these two lemmata, we are left to maximize a three-variable function parameterized by k and r . This is analogous to NAE k -SAT, except that there we only have to deal with a one-variable function, similarly parameterized. That simpler maximization, in fact, amounted to the bulk of the technical work in [2]. Luckily, here we will be able to get away with much less work: a convexity argument will allow us to reduce our three-dimensional optimization precisely to the optimization in [2].

2.1 Proof Outline for NAE k -SAT

Let Y be the number of satisfying assignments of a random NAE k -SAT formula with n variables and $m = rn$ clauses. It is easy to see that $\mathbf{E}[Y] = 2^n(1 - 2^{1-k})^{rn}$. Then $\mathbf{E}[Y^2]$ is the sum, over all ordered pairs of truth assignments, of the probability that both assignments in the pair are satisfying. It is not hard to show that if two assignments assign the same value to $z = \alpha n$ variables, then the probability that both are satisfying is

$$p(\alpha) = 1 - 2^{1-k} (2 - \alpha^k - (1 - \alpha)^k) .$$

Since there are $2^n \binom{n}{z}$ pairs of assignments sharing z variables, we have

$$\frac{\mathbf{E}[Y^2]}{\mathbf{E}[Y]^2} = \sum_{z=0}^n \binom{n}{z} \left[\frac{1}{2} \left(\frac{p(z/n)}{(1 - 2^{1-k})^2} \right)^r \right]^n .$$

To bound such sums within a constant factor, we proved the following in [2].

Lemma 2. *Let f be any real positive analytic function and let*

$$S = \sum_{z=0}^n \binom{n}{z} f(z/n)^n .$$

Define $0^0 \equiv 1$ and let g on $[0, 1]$ be

$$g(\alpha) = \frac{f(\alpha)}{\alpha^\alpha (1 - \alpha)^{1-\alpha}} .$$

If there exists $\alpha_{\max} \in (0, 1)$ such that $g(\alpha_{\max}) \equiv g_{\max} > g(\alpha)$ for all $\alpha \neq \alpha_{\max}$, and $g''(\alpha_{\max}) < 0$, then there exist constants B and C such that for all sufficiently large n

$$B \times g_{\max}^n \leq S \leq C \times g_{\max}^n .$$

Thus, using Lemma 2, bounding $\mathbf{E}[X^2]/\mathbf{E}[X]^2$ reduces to maximizing

$$\ell_r(\alpha) = \frac{1}{2\alpha^\alpha(1-\alpha)^{1-\alpha}} \left(\frac{p(\alpha)}{(1-2^{1-k})^2} \right)^r. \tag{3}$$

Note now that $\ell_r(1/2) = 1$ for all r and that our goal is to find r such that $\ell_r(\alpha) \leq 1$ for all $\alpha \in [0, 1]$. Indeed, in [2] we showed that

Lemma 3. [2] *For every $\epsilon > 0$, and all $k \geq k_0(\epsilon)$, if*

$$r \leq 2^{k-1} \ln 2 - \frac{\ln 2}{2} - \frac{1 + \epsilon}{2}$$

then $\ell_r(1/2) = 1 > \ell_r(\alpha)$ for all $\alpha \neq 1/2$ and $\ell_r''(1/2) < 0$.

Thus, for all r, k, ϵ as in Lemma 3, we see that Lemma 2 implies $\mathbf{E}[Y^2]/\mathbf{E}[Y]^2 < C \times \ell_r(1/2)^n = C$, concluding the proof.

3 Proof Outline for Hypergraph 2-Colorability

Let X be the number of 2-colorings of $H_k(n, rn)$. Let $q = 1 - 2^{1-k}$ and

$$p(\alpha, \beta, \gamma) = 1 - \alpha^k - (1-\alpha)^k - \beta^k - (1-\beta)^k + \gamma^k + (\alpha-\gamma)^k + (\beta-\gamma)^k + (1-\alpha-\beta+\gamma)^k.$$

We will prove that

Lemma 4. *There exists a constant A such that*

$$\frac{\mathbf{E}[X^2]}{\mathbf{E}[X]^2} \leq \frac{1}{A^2} \sum_{z_1+\dots+z_d=n} \binom{n}{z_1, z_2, z_3, z_4} \left(\frac{1}{4} \left(\frac{p\left(\frac{z_1+z_2}{n}, \frac{z_1+z_3}{n}, \frac{z_1}{n}\right)}{q^2} \right)^r \right)^n.$$

Similarly to NAE k -SAT we would like to bound this sum by a function of its maximum term. To do this we will establish a multidimensional generalization of the upper bound of Lemma 2.

Lemma 5. *Let f be any real positive analytic function and let*

$$S = \sum_{z_1+\dots+z_d=n} \binom{n}{z_1, \dots, z_d} f(z_1/n, \dots, z_{d-1}/n)^n.$$

Let $Z = \left\{ (\zeta_1, \dots, \zeta_{d-1}) : \zeta_i \geq 0 \text{ for all } i, \text{ and } \sum \zeta_i \leq 1 \right\}$. Define g on Z as

$$g(\zeta_1, \dots, \zeta_{d-1}) = \frac{f(\zeta_1, \dots, \zeta_{d-1})}{\zeta_1^{\zeta_1} \dots \zeta_{d-1}^{\zeta_{d-1}} (1 - \zeta_1 - \dots - \zeta_{d-1})^{1-\zeta_1-\dots-\zeta_{d-1}}}.$$

If *i)* there exists ζ_{\max} in the interior of Z such that for all $\zeta \in Z$ with $\zeta \neq \zeta_{\max}$, we have $g(\zeta_{\max}) \equiv g_{\max} > g(\zeta)$, and *ii)* the determinant of the $(d-1) \times (d-1)$ matrix of second derivatives of g is nonzero at ζ_{\max} , then there exists a constant D such that for all sufficiently large n

$$S < D \times g_{\max}^n.$$

Applying Lemma 5 to the sum in Lemma 4 we see that we need to maximize

$$g_r(\alpha, \beta, \gamma) = \frac{\left(\frac{p(\alpha, \beta, \gamma)}{q^2}\right)^r}{4\gamma^\gamma(\alpha - \gamma)^{\alpha - \gamma}(\beta - \gamma)^{\beta - \gamma}(1 - \alpha - \beta + \gamma)^{1 - \alpha - \beta + \gamma}} \quad (4)$$

where for convenience we defined g_r in terms of α, β, γ instead of $\zeta_1, \zeta_2, \zeta_3$. We will show that g_r has a unique maximum at

$$\zeta^* = (1/2, 1/2, 1/4) \ .$$

Lemma 6. *For every $\epsilon > 0$, and all $k \geq k_0(\epsilon)$ if*

$$r \leq 2^{k-1} \ln 2 - \frac{\ln 2}{2} - \frac{1 + \epsilon}{2}$$

then $g_r(\zeta^) = 1 > g_r(\zeta)$ for all $\zeta \in Z$ with $\zeta \neq \zeta^*$. Moreover, the determinant of the matrix of second derivatives of g_r at ζ^* is nonzero.*

Therefore, for all r, k, ϵ as in Lemma 6

$$\frac{\mathbf{E}[X^2]}{\mathbf{E}[X]^2} < \frac{D}{A^2} \times g_r(\zeta^*)^n = D/A^2 \ ,$$

completing the proof of Theorem 1 modulo Lemmata 4, 5 and 6.

The proof of Lemma 4 is a straightforward probabilistic calculation. The proof of Lemma 5 is somewhat technical but follows standard asymptotic methods. To prove Lemma 6 we will rely very heavily on Lemma 3. In particular, we will show that all local maxima of g_r occur within a one-dimensional subspace, in which g_r coincides with the function ℓ_r of (3). Specifically, we prove

Lemma 7. *If (α, β, γ) is a local extremum of g_r , then $\alpha = \beta = 1/2$.*

This reduces our problem to the one-dimensional maximization for NAE k -SAT, allowing us to easily prove Lemma 6.

Proof of Lemma 6. Observe that

$$g_r(1/2, 1/2, \gamma) = \ell_r(2\gamma) \ ,$$

where ℓ_r is the function defined in (3) for NAE k -SAT. Thus, the inequality $g_r(\zeta^*) > g_r(\zeta)$ for $\zeta \neq \zeta^*$ follows readily from Lemma 3, giving the first part of the lemma.

To prove the condition on the determinant of the 3×3 matrix of second derivatives, a little arithmetic shows that at ζ^* it is equal to

$$\frac{256(2^k - 2 - 2kr + 2k^2r)^2}{2^{4k}q^4} (4k(k - 1)r - 2^{2k}q^2) \ .$$

Thus, the determinant is negative whenever

$$4k(k - 1)r < 2^{2k}q^2 \ .$$

For $k = 3, 4$ this is true for $r < 3/2$ and $r < 49/12$ respectively, while for $k \geq 5$ it is true for all $r < \ln 2 \times 2^{k-1}$. □

4 Proof of Lemma 4

Recall that X denotes the number of 2-colorings of $H_k(n, m = rn)$.

4.1 First Moment

Recall that

$$q = 1 - 2^{1-k} .$$

The probability that a 2-coloring with $z = \alpha n$ black vertices and $n - z = (1 - \alpha)n$ white vertices makes a random hyperedge of size k bichromatic is

$$s(\alpha) = 1 - \alpha^k - (1 - \alpha)^k \leq q .$$

Summing over the 2^n colorings gives

$$\mathbf{E}[X] = \sum_{z=0}^n \binom{n}{z} s(z/n)^{rn} .$$

To bound this sum from below we apply the lower bound of Lemma 2 with $f(\alpha) = s(\alpha)^r$. In particular, it is easy to see that for all $r > 0$

$$g(\alpha) = \frac{s(\alpha)^r}{\alpha^\alpha(1-\alpha)^{1-\alpha}} = \frac{(1 - \alpha^k - (1 - \alpha)^k)^r}{\alpha^\alpha(1 - \alpha)^{1-\alpha}}$$

is maximized at $\alpha = 1/2$ and that $g(1/2) = 2q^r$. Moreover, for any $k > 1$

$$g''(1/2) = -8(1 - 2^{1-k})^{r-1} (1 + 2^{1-k}(k(k-1)r - 1)) < 0 .$$

Therefore, we see that there exists a constant A such that

$$\mathbf{E}[X] \geq A \times (2q^r)^n . \tag{5}$$

4.2 Second Moment

We first observe that $\mathbf{E}[X^2]$ equals the expected number of ordered pairs S, T of 2-partitions of the vertices such that both S and T are 2-colorings. Suppose that S and T have αn and βn black vertices respectively, while γn vertices are black in both. By inclusion-exclusion a random hyperedge of size k is bichromatic under both S and T with probability $p(\alpha, \beta, \gamma)$, i.e.

$$1 - \alpha^k - (1 - \alpha)^k - \beta^k - (1 - \beta)^k + \gamma^k + (\alpha - \gamma)^k + (\beta - \gamma)^k + (1 - \alpha - \beta + \gamma)^k .$$

The negative terms above represent the probability that the hyperedge is monochromatic under either S or T , while the positive terms represent the probability that it is monochromatic under both (potentially with different colors). Since the $m = rn$ hyperedges are chosen independently and with replacement, the probability that all $m = rn$ hyperedges are bichromatic is $p(\alpha, \beta, \gamma)^{rn}$.

If z_1, z_2, z_3 and z_4 vertices are respectively black in both assignments, black in S and white in T , white in S and black in T , and white in both, then $\alpha = (z_1 + z_2)/n$, $\beta = (z_1 + z_3)/n$ and $\gamma = z_1/n$. Thus,

$$\mathbf{E}[X^2] = \sum \binom{n}{z_1, z_2, z_3, z_4} p \left(\frac{z_1 + z_2}{n}, \frac{z_1 + z_3}{n}, \frac{z_3}{n} \right)^{rn}. \tag{6}$$

□

5 Proof of Lemma 7

We wish to show that at any extremum of g_r we have $\alpha = \beta = 1/2$. We start by proving that at any such extremum $\alpha = \beta$. Note that since, by symmetry, we are free to flip either or both colorings, we can restrict ourselves to the case where $\alpha \leq 1/2$ and $\gamma \leq \alpha/2$.

Let $h(x_1, x_2, x_3, x_4) = \sum_i x_i \ln x_i$ denote the entropy function, and let us define the shorthand $(\partial/\partial x - \partial/\partial y)f$ for $\partial f/\partial x - \partial f/\partial y$. Also, recall that $q = 1 - 2^{1-k}$ and that $p(\alpha, \beta, \gamma) \equiv p$ is

$$1 - \alpha^k - (1 - \alpha)^k - \beta^k - (1 - \beta)^k + \gamma^k + (\alpha - \gamma)^k + (\beta - \gamma)^k + (1 - \alpha - \beta + \gamma)^k.$$

We will consider the gradient of $\ln g_r$ along a vector that increases α while decreasing β . We see

$$\begin{aligned} & \left(\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta} \right) \ln g_r(\alpha, \beta, \gamma) \\ &= \left(\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta} \right) \left(h(\gamma, \alpha - \gamma, \beta - \gamma, 1 - \alpha - \beta + \gamma) - \ln 4 + r(\ln p - 2 \ln q) \right) \\ &= -\ln(\alpha - \gamma) + \ln(\beta - \gamma) \\ & \quad + \frac{kr}{p} \left(-\alpha^{k-1} + (1 - \alpha)^{k-1} + (\alpha - \gamma)^{k-1} + \beta^{k-1} - (1 - \beta)^{k-1} - (\beta - \gamma)^{k-1} \right) \\ &\equiv \phi(\alpha) - \phi(\beta), \end{aligned} \tag{7}$$

where

$$\phi(x) = -\ln(x - \gamma) + \frac{kr}{p} \left(-x^{k-1} + (1 - x)^{k-1} + (x - \gamma)^{k-1} \right).$$

Here we regard p as a constant in the definition of $\phi(x)$.

Observe now that if (α, β, γ) is an extremum of g_r then it is also an extremum of $\ln g_r$. Therefore, it must be that $(\partial/\partial \alpha) \ln g_r = (\partial/\partial \beta) \ln g_r = 0$ at (α, β, γ) which, by (7), implies $\phi(\alpha) = \phi(\beta)$. This, in turn, implies $\alpha = \beta$ since $\phi(x)$ is monotonically decreasing in the interval $\gamma < x < 1$:

$$\frac{d\phi}{dx} = -\frac{1}{x - \gamma} - \frac{k(k-1)r}{p} \left(x^{k-2} + (1 - x)^{k-2} - (x - \gamma)^{k-2} \right) < 0.$$

Next we wish to show that in fact $\alpha = \beta = 1/2$. Setting $\alpha = \beta$, we consider the gradient of $\ln g_r$ along a vector that increases α and γ simultaneously (using a similar shorthand for $\partial g/\partial\alpha + \partial g/\partial\gamma$):

$$\begin{aligned} & \left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\gamma} \right) \ln g_r(\alpha, \alpha, \gamma) \\ &= \left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\gamma} \right) \left(h(\gamma, \alpha - \gamma, \alpha - \gamma, 1 - 2\alpha + \gamma) - \ln 4 + r(\ln p - 2 \ln q) \right) \\ &= -\ln \gamma + \ln(1 - 2\alpha + \gamma) \\ &\quad + \frac{kr}{p} \left(-2\alpha^{k-1} + 2(1 - \alpha)^{k-1} - (1 - 2\alpha + \gamma)^{k-1} + \gamma^{k-1} \right) \\ &\equiv \psi(\alpha) . \end{aligned}$$

Clearly, $\psi(\alpha) = 0$ when $\alpha = 1/2$. To show that $1/2$ is the only such α , we show that ψ decreases monotonically with α by showing that if $0 < \alpha < 1/2$ and $\gamma \leq \alpha/2$, all three terms below are negative for $k \geq 3$.

$$\begin{aligned} \frac{\partial\psi}{\partial\alpha} &= -\frac{2}{1 - 2\alpha + \gamma} \\ &\quad + \frac{2k(k-1)r}{p} \left(-\alpha^{k-2} - (1 - \alpha)^{k-2} + (1 - 2\alpha + \gamma)^{k-2} \right) \\ &\quad + \frac{2k^2r}{p^2} \times \left(\gamma^{k-1} - (1 - 2\alpha + \gamma)^{k-1} - 2\alpha^{k-1} + 2(1 - \alpha)^{k-1} \right) \\ &\quad \times \left(-(\alpha - \gamma)^{k-1} + \alpha^{k-1} + (1 - 2\alpha + \gamma)^{k-1} - (1 - \alpha)^{k-1} \right) . \end{aligned}$$

The first and second terms are negative since $1 - \alpha > 1 - 2\alpha + \gamma > 0$, implying $(1 - \alpha)^{k-2} > (1 - 2\alpha + \gamma)^{k-2}$. The second factor of the third term is positive since $f(z) = z^{k-1}$ is convex and $(1 - \alpha) - \alpha = (1 - 2\alpha + \gamma) - \gamma$ (the factor of 2 on the last two terms only helps us since $1 - \alpha \geq \alpha$). Similarly, the third factor is negative since $(1 - \alpha) - (1 - 2\alpha + \gamma) = \alpha - \gamma \geq \alpha - (\alpha - \gamma) = \gamma$.

Thus, $\partial\psi/\partial\alpha < 0$ and $\alpha = 1/2$ is the unique solution to $\psi(\alpha) = 0$. Therefore, if (α, α, γ) is an extremum of g_r we must have $\alpha = 1/2$. \square

6 Proof of Lemma 5

6.1 Preliminaries

We will use the following form of Stirling's approximation for $n!$, valid for $n > 0$

$$\sqrt{2\pi n} n^n e^{-n} \left(1 + \frac{1}{12n} \right) < n! < \sqrt{2\pi n} n^n e^{-n} \left(1 + \frac{1}{6n} \right) . \quad (8)$$

We will also use the following crude lower bound for $n!$, valid for $n \geq 0$

$$n! \geq (n/e)^n , \quad (9)$$

using the convention $0^0 \equiv 1$.

Let z_1, \dots, z_d be such that $\sum_{i=1}^d z_i = n$. Let $\zeta_i = z_i/n$. Let $\zeta = (\zeta_1, \dots, \zeta_{d-1})$.

- If $z_i > 0$ for all i , then using the upper and lower bounds of (8) for $n!$ and $z_i!$ respectively, and reducing the denominator further by changing the factor $1 + 1/(12z_i)$ to 1 for $i \neq 1$, we get

$$\begin{aligned} \binom{n}{z_1, \dots, z_d} &< (2\pi n)^{-(d-1)/2} \left(\prod_{i=1}^d \zeta_i^{-1/2} \right) \left(\prod_{i=1}^d \zeta_i^{-\zeta_i} \right)^n \times \frac{1 + 1/(6n)}{1 + 1/(12z_1)} \\ &\leq (2\pi n)^{-(d-1)/2} \left(\prod_{i=1}^d \zeta_i^{-1/2} \right) \left(\prod_{i=1}^d \zeta_i^{-\zeta_i} \right)^n, \end{aligned} \tag{10}$$

where for (10) we assumed w.l.o.g. that $z_1 \leq n/2$. Thus,

$$\binom{n}{z_1, \dots, z_d} f(z_1/n, \dots, z_{d-1}/n)^n \leq (2\pi n)^{-(d-1)/2} \left(\prod_{i=1}^d \zeta_i^{-1/2} \right) g(\zeta)^n. \tag{11}$$

- For any $z_i \geq 0$, the upper bound of (8) and (9) give

$$\binom{n}{z_1, \dots, z_d} < \frac{7}{6} \sqrt{2\pi n} \left(\prod_{i=1}^d \zeta_i^{-\zeta_i} \right)^n,$$

implying a cruder bound

$$\binom{n}{z_1, \dots, z_d} f(z_1/n, \dots, z_{d-1}/n)^n \leq \frac{7}{6} \sqrt{2\pi n} g(\zeta)^n. \tag{12}$$

6.2 The Main Proof

Our approach is a crude form of the Laplace method for asymptotic integrals [9] which amounts to approximating functions near their peak as Gaussians.

We wish to approximate $g(\zeta)$ in the vicinity of ζ_{\max} . We will do this by Taylor expanding $\ln g$, which is analytic since g is analytic and positive. Since $\ln g$ increases monotonically with g , both g and $\ln g$ are maximized at ζ_{\max} . Furthermore, at ζ_{\max} the matrix of second derivatives of $\ln g$ is that of g divided by a constant, since

$$\left. \frac{\partial^2 \ln g}{\partial \zeta_i \partial \zeta_j} \right|_{\zeta = \zeta_{\max}} = \frac{1}{g_{\max}} \frac{\partial^2 g}{\partial \zeta_i \partial \zeta_j} - \frac{1}{g_{\max}^2} \frac{\partial g}{\partial \zeta_i} \frac{\partial g}{\partial \zeta_j}$$

and at ζ_{\max} the first derivatives of g are all zero. Therefore, if the matrix of second derivatives of g at ζ_{\max} has nonzero determinant, so does the matrix of the second derivatives of $\ln g$.

Note now that since the matrix of second derivatives is by definition symmetric, it can be diagonalized, and its determinant is the product of its eigenvalues. Therefore, if its determinant is nonzero, all its eigenvalues are smaller than some $\lambda_{\max} < 0$. Thus, Taylor expansion around ζ_{\max} gives

$$\ln g(\zeta) \leq \ln g_{\max} + \frac{1}{2} \lambda_{\max} |\zeta - \zeta_{\max}|^2 + O(|\zeta - \zeta_{\max}|^3)$$

or, exponentiating to obtain g ,

$$g(\zeta) \leq g_{\max} \exp\left(\frac{1}{2}\lambda_{\max}|\zeta - \zeta_{\max}|^2\right) \times (1 + O(|\zeta - \zeta_{\max}|^3)) .$$

Therefore, there is a ball of radius $\rho > 0$ around ζ_{\max} and constants $Y > 0$ and $g_* < g_{\max}$ such that

$$\text{If } |\zeta - \zeta_{\max}| \leq \rho, \quad g(\zeta) \leq g_{\max} \exp(-Y|\zeta - \zeta_{\max}|^2) , \quad (13)$$

$$\text{If } |\zeta - \zeta_{\max}| > \rho, \quad g(\zeta) \leq g_* . \quad (14)$$

We will separate S into two sums, one inside the ball and one outside:

$$\sum_{\zeta \in Z: |\zeta - \zeta_{\max}| \leq \rho} \binom{n}{\zeta_{1n}, \dots, \zeta_{dn}} f(\zeta)^n + \sum_{\zeta \in Z: |\zeta - \zeta_{\max}| > \rho} \binom{n}{\zeta_{1n}, \dots, \zeta_{dn}} f(\zeta)^n .$$

For the terms inside the ball, first note that if $|\zeta - \zeta_{\max}| \leq \rho$ then

$$\prod_{i=1}^d \zeta_i^{-1/2} \leq W \text{ where } W = \left(\min_i \zeta_{\max,i} - \rho\right)^{-d/2} .$$

Then, since $|\zeta - \zeta_{\max}|^2 = \sum_{i=1}^{d-1} (\zeta_i - \zeta_{\max,i})^2$, using (11) and (13) we have

$$\begin{aligned} & \sum_{\zeta \in Z: |\zeta - \zeta_{\max}| \leq \rho} \binom{n}{\zeta_{1n}, \dots, \zeta_{dn}} f(\zeta)^n \\ & \leq (2\pi n)^{-(d-1)/2} W g_{\max}^n \times \sum_{z_1, \dots, z_{d-1} = -\infty}^{\infty} \exp\left(-nY \sum_{i=1}^{d-1} (\zeta_i - \zeta_{\max,i})^2\right) \\ & = (2\pi n)^{-(d-1)/2} W g_{\max}^n \times \prod_{i=1}^{d-1} \left(\sum_{z_i = -\infty}^{\infty} \exp(-nY (z_i/n - \zeta_{\max,i})^2) \right) . \end{aligned}$$

Now if a function $\phi(z)$ has a single peak, on either side of which it is monotonic, we can replace its sum with its integral with an additive error at most twice its largest term:

$$\left| \sum_{z=-\infty}^{\infty} \phi(z) - \int_{-\infty}^{\infty} \phi(z) dz \right| \leq 2 \max_z \phi(z)$$

and so

$$\begin{aligned} \sum_{z_i = -\infty}^{\infty} \exp(-nY(z_i/n - \zeta_{\max,i})^2) & \leq 2 + \int_{-\infty}^{\infty} \exp(-nY(z_i/n - \zeta_{\max,i})^2) dz \\ & = \sqrt{\pi n/Y} + 2 < \sqrt{2\pi n/Y} \end{aligned}$$

where the last inequality holds for sufficiently large n . Multiplying these $d - 1$ sums together gives

$$\sum_{\zeta \in Z: |\zeta - \zeta_{\max}| \leq \rho} \binom{n}{\zeta_{1n}, \dots, \zeta_{dn}} f(\zeta)^n \leq WY^{-(d-1)/2} g_{\max}^n . \quad (15)$$

Outside the ball, we use (12), (14) and the fact that the entire sum has at most n^{d-1} terms to write

$$\sum_{\zeta \in Z: |\zeta - \zeta_{\max}| > \rho} \binom{n}{\zeta_1 n, \dots, \zeta_d n} f(\zeta)^n \leq n^{d-1} \times \frac{7}{6} \sqrt{2\pi n} g_*^n < g_{\max}^n \quad (16)$$

where the last inequality holds for sufficiently large n . Combining (16) and (15) gives

$$S < (WY^{-(d-1)/2} + 1) g_{\max}^n \equiv D \times g_{\max}^n$$

which completes the proof. (We note that the constant D can be optimized by replacing our sums by integrals and using Laplace’s method [2,9].)

7 Conclusions

We have shown that the second moment method yields a very sharp estimate of the threshold for hypergraph 2-colorability. It allows us not only to close the asymptotic gap between the previously known bounds but, in fact, to get the threshold within a small additive constant. Yet:

- While the second moment method tells us that w.h.p. an exponential number of 2-colorings exist for $r = \Theta(2^k)$, it tells us nothing about how to find a single one of them efficiently. The possibility that such colorings actually cannot be found efficiently is extremely intriguing.
- While we have shown that the second moment method works really well, we’d be hard pressed to say why. In particular, we do not have a criterion for determining a constraint satisfaction problem’s amenability to the method. The fact that the method fails spectacularly for random k -SAT suggests that, perhaps, rather subtle forces are at play.

Naturally, one can always view the success of the second moment method in a particular problem as an a posteriori indication that the satisfying solutions of the problem are “largely uncorrelated”. This viewpoint, though, is hardly predictive. (Yet, it might prove useful to the algorithmic question above).

The solution-symmetry shared by NAE k -SAT and hypergraph 2-colorability but not by k -SAT, i.e. the property that the complement of a solution is also a solution, explains why the method gives a nonzero lower bound for these two problems (and why it fails for k -SAT). Yet symmetry alone does not explain why the bound becomes essentially tight as k grows. In any case, we hope (and, worse, consider it natural) that an appropriate notion of symmetry is present in many more problems.

References

1. ACHLIOPTAS, D., KIM, J.H., KRIVELEVICH, M. AND TETALI, P. Two-coloring random hypergraphs. *Random Structures Algorithms*, 20(2):249–259, 2002.

2. ACHLIOPTAS, D., AND MOORE, C. The asymptotic order of the random k -SAT threshold. To appear in *43rd Annual Symposium on Foundations of Computer Science (Vancouver, BC, 2002)*.
3. ALON, N., AND SPENCER, J. A note on coloring random k -sets. Unpublished manuscript.
4. BECK, J. On 3-chromatic hypergraphs. *Discrete Math.* 24, 2 (1978), 127–137.
5. BERGE, C. *Hypergraphs*. North-Holland Publishing Co., Amsterdam, 1989. Combinatorics of finite sets, Translated from the French.
6. BERNSTEIN, F. Zur theorie der trigonometrische reihen. *Leipz. Ber.* 60, (1908), 325–328.
7. CHAO, M.-T., AND FRANCO, J. Probabilistic analysis of a generalization of the unit-clause literal selection heuristics for the k -satisfiability problem. *Inform. Sci.* 51, 3 (1990), 289–314.
8. CHVÁTAL, V., AND REED, B. Mick gets some (the odds are on his side). In *33rd Annual Symposium on Foundations of Computer Science (Pittsburgh, PA, 1992)*. IEEE Comput. Soc. Press, Los Alamitos, CA, 1992, pp. 620–627.
9. DE BRUIJN, N.G. *Asymptotic Methods in Analysis*. North-Holland, 1958.
10. ERDŐS, P. On a combinatorial problem. *Nordisk Mat. Tidskr.* 11 (1963), 5–10, 40.
11. ERDŐS, P., AND LOVÁSZ, L. Problems and results on 3-chromatic hypergraphs and some related questions. 609–627. *Colloq. Math. Soc. János Bolyai*, Vol. 10.
12. FLAJOLET, P., KNUTH, D. E., AND PITTEL, B. The first cycles in an evolving graph. *Discrete Math.* 75, 1-3 (1989), 167–215. *Graph theory and combinatorics* (Cambridge, 1988).
13. FRIEDGUT, E. Necessary and sufficient conditions for sharp thresholds of graph properties, and the k -SAT problem. *J. Amer. Math. Soc.* 12 (1999), 1017–1054.
14. FRIEZE, A. AND MCDIARMID, C. Algorithmic theory of random graphs. *Random Struct. Alg.* 10, (1997), 5–42.
15. KAROŃSKI, M. AND ŁUCZAK, T. Random hypergraphs. *Combinatorics, Paul Erdős is eighty*, Vol. 2 (Kesztheley, 1993), 283–293, *Bolyai Soc. Math. Stud.*, 2, Janos Bolyai Math. Soc., Budapest, 1996.
16. KRIVELEVICH, M. AND SUDAKOV, B. The chromatic numbers of random hypergraphs. *Random Struct. Alg.* 12, (1998), 381–403.
17. LOVÁSZ, L. Coverings and coloring of hypergraphs. *Proceedings of the Fourth Southeastern Conference on Combinatorics, Graph Theory, and Computing* (Boca Raton, Florida, 1973), 3–12.
18. MOTWANI, R. AND RAGHAVAN, P. *Randomized Algorithms*. Cambridge University Press, Cambridge, 1995.
19. RADHAKRISHNAN, J., AND SRINIVASAN, A. Improved bounds and algorithms for hypergraph 2-coloring. *Random Structures Algorithms* 16, 1 (2000), 4–32.
20. SCHMIDT-PRUZAN, J., AND SHAMIR, E., AND UPFAL, E. Random hypergraph coloring algorithms and the weak chromatic number. *Journal of Graph Theory* 8, (1985), 347–362.